**M308I Test # 2b** Nov 28, 2007 **SOLUTIONS:**

**(1)** Calculate the determinant of the matrix $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 9 & 3 & 6 \\ 2 & 4 & -2 & 5 \\ 1 & 7 & 6 & 6 \end{bmatrix}$.

$$A \rightsquigarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 5 & 5 & 4 \\ 0 & 0 & 0 & 3 \\ 0 & 5 & 7 & 5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 5 & 5 & 4 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 5 & 5 & 4 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} = T.$$

$\det(T) = 1 \cdot 5 \cdot 2 \cdot 3 = 30.$ There was one row interchange, so $\det(A) = -30$.

**(2)** Let $T$ be the linear transformation of the plane to itself for which $T(1,3) = (2,-1)$ and $T(2,5) = (4,1)$. Find $T(1,1)$.

We want constants $c_1$ and $c_2$ so that $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot c_1 + \begin{bmatrix} 2 \\ 5 \end{bmatrix} \cdot c_2$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 5 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 2 \end{array}\right], \quad \text{so } c_1 = -3, \quad c_2 = 2.$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot (-3) + \begin{bmatrix} 2 \\ 5 \end{bmatrix} \cdot 2$$

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \cdot (-3) + \begin{bmatrix} 4 \\ 1 \end{bmatrix} \cdot 2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

**(3)** $v_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}$

**(3a)** Are $v_1, v_2, v_3$ linearly independent? Explain your answer.

We ask if there is a non-zero solution to $v_1c_1 + v_2c_2 + v_3c_3 = 0$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 3 & -1 & 7 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$ So $c_3 = 1$, $c_2 = 1$, $c_1 = -2$. Thus,

$$\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \cdot (-2) + \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot 1 + \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} \cdot 1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This shows that $v_1$, $v_2$, $v_3$ are linearly dependent.

**(3b)** Are $v_1, v_2, v_4$ linearly independent? Explain your answer.

We ask if there is a non-zero solution to $v_1c_1 + v_2c_2 + v_4c_3 = 0$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 3 & -1 & 7 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$ The only solution is $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.

This says that $v_1$, $v_2$, $v_4$ are linearly independent.

**(3c)** Do the vectors $v_1, v_3, v_4$ span $R^3$? Explain your answer.

Suppose $w = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$. Then $[A|w] = [v_1v_2v_3|w] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & a_1 \\ 3 & 7 & 6 & a_2 \\ 2 & 2 & 2 & a_3 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{array}\right].$

Where $c_1$, $c_2$, and $c_3$ are whatever the right-most column becomes as we row-reduce $A$. This shows that there always is a solution to $w = v_1c_1 + v_3c_2 + v_4c_3$, so $v_1, v_3, v_4$ do span $R^3$.