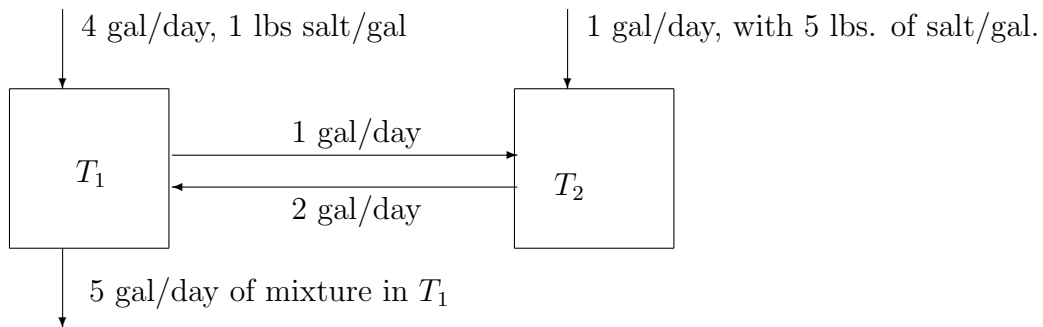


M309 Quiz # 1 Oct 8, 2007 **SOLUTIONS:**

(1a) There are two tanks, with pipes carrying salt water as shown:



T_1 holds 10 gallons of salt water; T_2 holds 20 gallons of salt water.

Let x_1 be pounds of salt in T_1 at time t ;

Let x_2 be pounds of salt in T_2 at time t ;

Write a system of differential equations for x_1 and x_2 .

SOLUTION:
$$\begin{cases} x_1' = -0.6x_1 + 0.1x_2 + 4 \\ x_2' = 0.1x_1 - 0.1x_2 + 5 \end{cases}$$

(1b) How many lbs of salt are in T_1 and T_2 when the system is in equilibrium?

SOLUTION: $x_1' = 0$ and $x_2' = 0$, So

$$\begin{aligned} -0.6x_1 + 0.1x_2 &= -4 \\ 0.1x_1 - 0.1x_2 &= -5 \end{aligned}$$

$$\begin{aligned} -0.5x_1 + 0 \cdot x_2 &= -9 \\ x_1 &= 18 \\ x_2 &= 68 \end{aligned}$$

(2a) Find the solution to $\begin{cases} x_1' = 3x_1 - 2x_2 \\ x_2' = 2x_1 - 2x_2 \end{cases}$ where $x(0) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

Solution: $\det \begin{bmatrix} 3 - \lambda & -2 \\ 2 & -2 - \lambda \end{bmatrix} = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$

For the eigenvalue $\lambda_1 = -1$, the eigenvector is $\xi_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

For the eigenvalue $\lambda_2 = +2$, the eigenvector is $\xi_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

The general solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}$

If $x(0) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, then $C_1 = 3$, $C_2 = -1$. So $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}$

(1b) The quadrants are numbered in the diagram below. The trajectory of **(1b)** starts in quadrant *I*. In which quadrant (*I*, *II*, *III* or *IV*) will the trajectory be when $t = 1000$?

Answer: $\begin{bmatrix} x_1(1000) \\ x_2(1000) \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \epsilon + \begin{bmatrix} -2 \\ -1 \end{bmatrix} B$, where ϵ is very small and B is very big.

The trajectory starts in *I*, then goes to *II*, and finally goes into *III*, where it remains.

