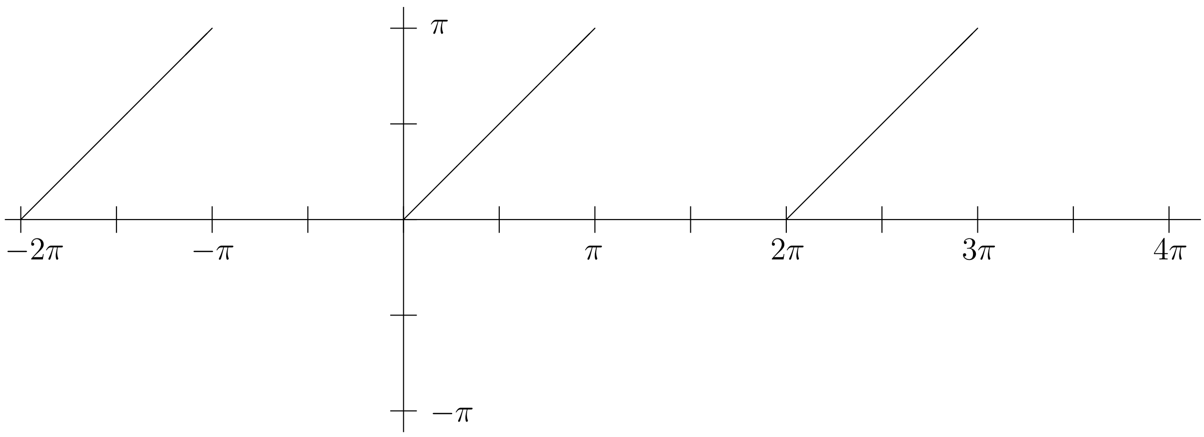
M309 Sample Problems for Test # 2 Feb 23, 2007 (with some Solutions).

(1) The temperature in a rod of length π satisfies the heat equation $u_t = 4u_{xx}$. The temperature initially is $u(x,0) = \sin x + \sin 5x$ for $0 \leq x \leq \pi$, and the temperature at each end is fixed at 0. Find the temperature $u(x,t)$ for all $0 \leq x \leq \pi$, and all $t \geq 0$.

SOLUTION $u(x,t) = \sin x \cdot e^{-4t} + \sin 5x \cdot e^{-25\cdot 4t}$

(2) $f(x) = \left\{ \begin{array}{ll} 0, & \text{for } -\pi \leq x < 0 \\ x, & \text{for } 0 \leq x < \pi \end{array} \right\}$ $f(x)$ is extended periodically with period 2π.

Sketch the graph of f for $-2\pi \leq x \leq 4\pi$. Calculate the Fourier Series for $f(x)$.



Solution: This problen is very similar to §10.2 #15 in text.

$$2\pi\frac{a_0}{2} = \int_0^{\pi} x dx = \frac{\pi^2}{2}; \quad \frac{a_0}{2} = \frac{\pi}{4}$$

$$\pi a_m = \int_0^{\pi} x \cos mx dx = \left[\frac{x \sin mx}{m} + \frac{\cos mx}{m^2}\right]_0^{\pi} = \left\{ \begin{array}{l} -\frac{2}{m^2} \text{ if m is odd} \\ 0, \text{ if m is even} \end{array} \right.$$

$$\pi b_m = \int_0^{\pi} x \sin mx dx = \left[-\frac{x \cos mx}{m} + \frac{\sin mx}{m^2}\right]_0^{\pi} = \left\{ \begin{array}{l} -\frac{\pi}{m} \text{ if m is odd} \\ \frac{\pi}{m}, \text{ if m is even} \end{array} \right.$$

$$f(x) = \frac{\pi}{4} + \sum_{k=0}^{\infty} \frac{-2}{\pi(2k+1)^2} \cos(2k+1)x + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \sin mx$$

(3) The temperature in a rod of length π satisfies the heat equation $u_t = 4u_{xx}$, and the temperature at each end is fixed at 0. The initial temperature is $u(x,0) = \begin{cases} x, & \text{for } 0 \le x < \frac{\pi}{2} \\ \pi - x, & \text{for } \frac{\pi}{2} \le x < \pi \end{cases}$ Find the temperature $u(x,t)$ for all $0 \le x \le \pi$, and all $t \ge 0$.

Solution: First calculate the Fourier Sine Series for the function $f(x) = u(x,0)$.

$$\pi b_n = 2\int_0^{\pi} f(x)\sin nx dx = 2\Big[\int_0^{\frac{\pi}{2}} x\sin nx dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x)\sin nx dx\Big]$$

For n even, the two integrals are equal, but with opposits signs, $b_{2k} = 0$

For n odd, the two integrals are equal, with same sign. Therefore, for n odd:

$$\pi b_n = 2\int_0^{\pi} f(x)\sin nx dx = 4\int_0^{\frac{\pi}{2}} x\sin nx dx = 4\Big[-\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\Big]_0^{\frac{\pi}{2}} = 4\Big[\frac{\pm 1}{n^2}\Big]$$

The sign is $+1$ if $n = 1,\ 5,\ \ldots$.

The sign is -1 if $n = 3,\ 7,\ \ldots$.

Therefore, $f(x) = \dfrac{4}{\pi}\displaystyle\sum_{k=0}^{k=\infty} \frac{(-1)^k}{(2k+1)^2}\sin(2k+1)x$

The solution of the heat eqn is $u(x,t) = \dfrac{4}{\pi}\displaystyle\sum_{k=0}^{k=\infty} \frac{(-1)^k}{(2k+1)^2}\sin(2k+1)x e^{-4(2k+1)^2 t}$

(4) The temperature in a rod of length π satisfies the heat equation $u_t = 9u_{xx}$. The temperature initially is $u(x,0) = \begin{cases} 1, & \text{for } 0 \le x < \frac{\pi}{2} \\ 0, & \text{for } \frac{\pi}{2} \le x < \pi \end{cases}$ and the ends are insulated. Find the temperature $u(x,t)$ for all $0 \le x \le \pi$, and all $t \ge 0$.