M309 Test # 1a Oct 24, 2007 **SOLUTIONS:**

(1) (50 pts) Find the general solution to
$$\begin{aligned} x_1' &= 2x_1 - x_2 + e^t \\ x_2' &= -x_1 + 2x_2 - e^t \end{aligned}$$

SOLUTION: $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$$\det(A - rI) = \det \begin{bmatrix} 2-r & -1 \\ -1 & 2-r \end{bmatrix} = r^2 - 4r + 3 = (r-1)(r-3)$$

$$r = 1,\ A - I = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \xi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad r = 3,\ A - 3I = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}, \xi_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$$T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}; \quad T^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

$$y' = Dy + h, \text{ where } D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \ h = T^{-1}g = \frac{1}{2}\begin{bmatrix} 0 \\ -e^t \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} y_1 + 0 \\ 3y_2 - e^t \end{bmatrix},$$

$y_1' = y_1$, So $y_1 = C_1 e^t$

$y_2' = 4y_1 - e^t$, So $y_2 = \frac{1}{3}e^t + C_2 e^{3t}$

$$\text{Then } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} C_1 e^t \\ \frac{1}{3}e^t + C_2 e^{3t} \end{bmatrix} = \begin{bmatrix} C_1 e^t - \frac{1}{3}e^t - C_2 e^{3t} \\ C_1 e^t + \frac{1}{3}e^t + C_2 e^{3t} \end{bmatrix}$$

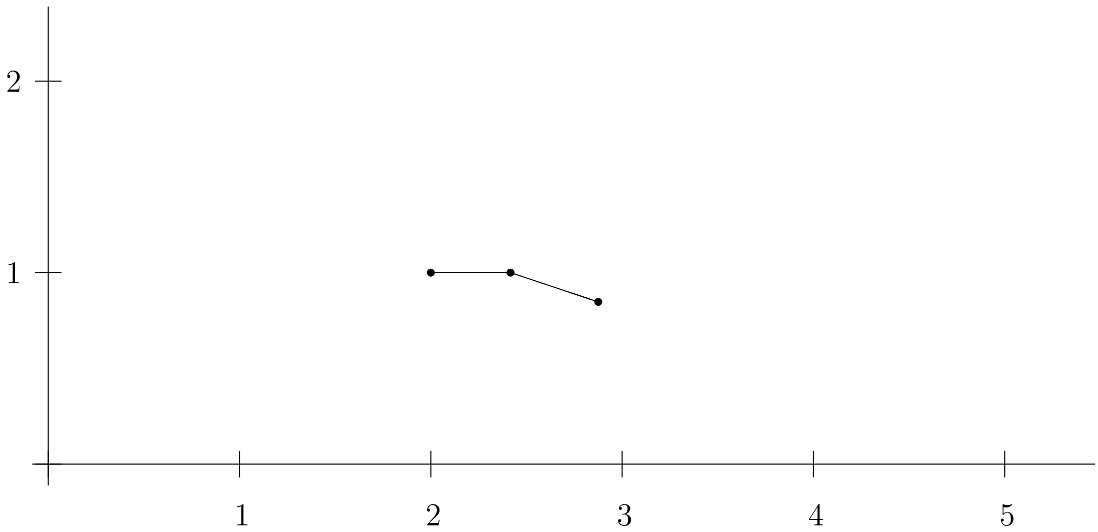
(2) We want an approximate solution to: $\begin{array}{l} x' = x(y-1) \\ y' = y(2-x) \end{array}$ with $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

(2a) (20 pts) Do two steps, each with $\Delta t = 0.2$ to find approx. values for $\begin{bmatrix} x \\ y \end{bmatrix}$ at $t = 0.2$ and $t = 0.4$

SOLUTION: At $(x_0 = 2, y_0 = 2)$, $x' = 2,\ \Delta x \approx 0.4,\ x_1 \approx 2.4$
$y' = 0,\ \Delta y = 0,\ y_1 \approx 2$

At $(x_1 = 2.4, y_1 = 2)$, $x' = 2.4,\ \Delta x \approx 0.48,\ x_2 \approx 2.88$
$y' = -.8,\ \Delta y = -.16,\ y_2 \approx 1.84$

(2b) (10 pts) Plot this (piecewise linear) partial trajectory for $0 \le t \le 0.4$.



(2c) (20 pts) The point $(2, 1)$ is a critical point of the system of **(2a)**. What is the type and stability of this point?

SOLUTION: Take $u = x - 2$, $y = v - 2$.

$$\begin{array}{r} u' = (u+2)v = uv + 2v \approx 2v \\ v' = v + 1(-u) = -uv - u \approx -u \end{array} \qquad A = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}.$$

$r^2 + 2 = 0;$, $r = \pm i\sqrt{2}$, pure imaginary.

$(2, 1)$ is a CENTER POINT; STABLE. Trajectories are ellipses.