M309 Test # 2 Feb 23, 2007 **SOLUTIONS**

(1) The temperature in a rod of length π satisfies the heat equation $u_t = 4u_{xx}$. The temperature initially is $u(x,0) = \sin 2x + \sin 5x$ for $0 \le x \le \pi$, and the temperature at each end is fixed at 0. Find the temperature $u(x,t)$ for all $0 \le x \le \pi$, and all $t \ge 0$.

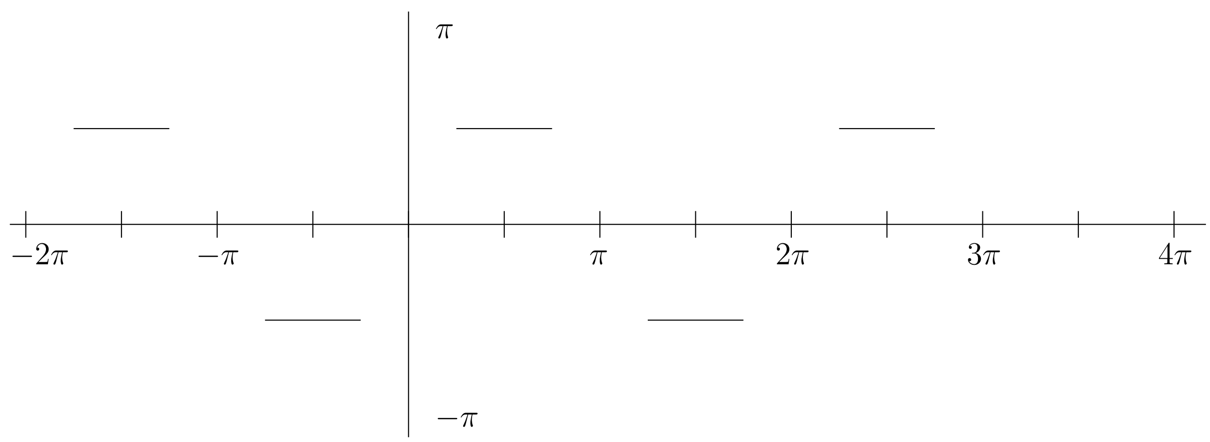
SOLUTION $u(x,t) = \sin 2x \cdot e^{-16t} + \sin 5x \cdot e^{-100t}$

(2) The temperature in a rod of length π satisfies the heat equation $u_t = 9u_{xx}$. The temperature initially is $u(x,0) = 1 + \cos x + \cos 2x$ for $0 \le x \le \pi$, and the ends are insulated. Find the temperature $u(x,t)$ for all $0 \le x \le \pi$, and all $t \ge 0$.

SOLUTION $u(x,t) = 1 + \cos x \cdot e^{-9t} + \cos 2x \cdot e^{-36t}$

(3) $f(x) = \left\{ \begin{array}{lll} 0, & \text{for} & 0 \le x < \frac{\pi}{4} \\ \frac{\pi}{2}, & \text{for} & \frac{\pi}{4} \le x < \frac{3\pi}{4} \\ 0, & \text{for} & \frac{3\pi}{4} \le x < \pi \end{array} \right\}$ $f(x)$ is extended as an odd function for $-\pi \le x \le \pi$.

Then $f(x)$ is extended periodically with period 2π. Sketch the graph of f for $-2\pi \le x \le 4\pi$.



(4) $f(x) = \left\{ \begin{array}{lll} 0, & \text{for} & -\pi \leq x < 0 \\ 1, & \text{for} & 0 \leq x \leq \frac{\pi}{2} \\ 0, & \text{for} & \frac{\pi}{2} < x < \pi \end{array} \right\}$ $\quad$ $f(x)$ is extended periodically with period 2π.

Calculate the Fourier Series for $f(x)$.

SOLUTION:

$$\frac{a_0}{2} = \frac{1}{4}$$

$$\pi a_n = \int_0^{\frac{\pi}{2}} \cos nx dx = \left[\frac{1}{n} \sin nx\right]_0^{\frac{\pi}{2}} = \left\{ \begin{array}{lll} 0, & \text{for} & n \text{ even} \\ \pm\frac{1}{n}, & \text{for} & n \text{ odd} \end{array} \right\}$$

$$\pi b_n = \int_0^{\frac{\pi}{2}} \sin nx dx = \left[-\frac{1}{n} \cos nx\right]_0^{\frac{\pi}{2}} = \frac{1}{n}\Big(-\cos(\frac{n\pi}{2}) + 1\Big)$$

$$\pi a_{2k+1} = \frac{(-1)^k}{2k+1}$$

$$\pi b_n = \frac{1}{n}\Big(-\cos(\frac{n\pi}{2}) + 1\Big)$$

$$\pi b_{2k} = \left\{ \begin{array}{lll} \frac{1}{2k}, & \text{for} & k \text{ odd} \\ 0 & \text{for} & k \text{ even} \end{array} \right\}$$

$$f(x) = \frac{1}{4} + \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos(2k+1)x + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{1}{n}\Big(-\cos(\frac{n\pi}{2}) + 1\Big) \sin nx$$