M309 Quiz # 5 Dec 1, 2000 Name ....

(1) A metal rod of length π is insulated so that no heat flows into or out of the ends (or sides). The temperature $u(x,t)$ satisfies the heat equation $u_t = u_{xx}$. The initial temperature is $u(x,0) = x$ for $0 \le x < \pi$. Find the temperature $u(x,t)$ for all $0 \le x \le \pi$, and all $t \ge 0$.

SOLUTION Let $f(x) = u(x,0)$; $f(x)$ is extended as an <u>even</u> function for $-\pi \le x \le \pi$.

Calculate the Fourier Cosine Series for $f(x)$:

$$2\pi \frac{a_0}{2} = 2\int_0^{\pi} x dx = 2\frac{\pi^2}{2} = \pi^2 \qquad \frac{a_0}{2} = \frac{\pi}{2}$$

$$\pi a_n = 2\int_0^{\pi} f(x)\cos nx dx \quad = 2\int_0^{\pi} x\cos nx dx$$

$$= 2\Big[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\Big]_0^{\pi} = \frac{2}{n^2}\Big[\cos n\pi - 1\Big]$$

$$a_n = -\frac{4}{\pi n^2} \quad \text{if } n \text{ is odd}$$

$$a_n = 0 \qquad \text{if } n \text{ is even}$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{k=\infty} \frac{1}{(2k+1)^2}\cos\Big((2k+1)x\Big)$$

The solution of the heat equation is: $u(x,t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{k=\infty} \frac{1}{(2k+1)^2}\cos\Big((2k+1)x\Big)e^{-(2k+1)^2 t}$