

(1a) Find the general (real) solution to $x' = Ax$, where $A = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix}$

Solution

The characteristic poly is $\lambda^2 + 2\lambda + 2$. The eigenvalues are $\lambda = -1 \pm i$

For $\lambda = -1 + i$, solving $\begin{bmatrix} 2-i & -1 \\ 5 & -2-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$; $\xi = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2-i \end{bmatrix}$

The cplx solution is $\begin{bmatrix} 1 \\ 2-i \end{bmatrix} e^{-t}(\cos t + i \sin t) = \begin{bmatrix} \cos t + i \sin t \\ 2 \cos t + \sin t + i(2 \sin t - \cos t) \end{bmatrix} e^{-t}$

The Real and Imaginary parts are $\begin{bmatrix} \cos t \\ 2 \cos t + \sin t \end{bmatrix} e^{-t}$ and $\begin{bmatrix} \sin t \\ 2 \sin t - \cos t \end{bmatrix} e^{-t}$

The general (real) solution is $x = C_1 \cdot \begin{bmatrix} \cos t \\ 2 \cos t + \sin t \end{bmatrix} e^{-t} + C_2 \cdot \begin{bmatrix} \sin t \\ 2 \sin t - \cos t \end{bmatrix} e^{-t}$

There are equivalent solutions arising from a different choice of ξ .

(1b) Find the solution to (1a) for which $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Solution

$C_1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$; $C_1 = 2$; $C_2 = 1$.

(1b') Some of the tests had $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, in which case $C_1 = 3$, $C_2 = 4$.

(2) Find the general solution to $x' = Ax + g$, $A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$, $g = \begin{bmatrix} 3e^t \\ 0 \end{bmatrix}$.

Solution The char. poly is $\lambda^2 - 5\lambda + 4$. The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 4$.

The eigenvectors are $\xi_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\xi_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. $T = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$, $T^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$

Then $AT = TD$. where $D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$; also $h = T^{-1}g = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \cdot g = \begin{bmatrix} 2e^t \\ e^t \end{bmatrix}$.

$$\begin{aligned} x' &= Ax + g \\ x' &= TDT^{-1}x + g \\ T^{-1}x' &= DT^{-1}x + t^{-1}g \\ y' &= Dy + h \end{aligned}$$

$$\begin{aligned} y_1' &= y_1 + 2e^t \\ y_1' - y_1 &= 2e^t \\ e^{-t}(y_1' - y_1) &= 2 \\ e^{-t}(y_1) &= 2t + C_1 \\ y_1 &= 2te^t + C_1e^t \end{aligned}$$

$$\begin{aligned} y_2' &= 4y_2 + e^t \\ y_2' - 4y_2 &= e^t \\ e^{-4t}(y_2' - 4y_2) &= e^{-3t} \\ e^{-4t}(y_2) &= -\frac{1}{3}e^{-3t} + C_2 \\ y_2 &= -\frac{1}{3}e^t + C_2e^{4t} \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 + y_2 \\ -y_1 + 2y_2 \end{bmatrix}.$$

(2') Solve $x' = Ax + g$, where $A = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$, $g = \begin{bmatrix} 5e^t \\ 0 \end{bmatrix}$.

The solution is slightly different. The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 6$.

(Extra Credit) (EC a)

$A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$. $g = \begin{bmatrix} e^t \\ 0 \end{bmatrix}$. Find an invertible matrix T and a matrix J in Jordan form with $AT = TJ$. Find the general solution to $x' = Ax + g$, where $g = \begin{bmatrix} e^t \\ 0 \end{bmatrix}$.

Solution The characteristic polynomial is $\lambda^2 - 4\lambda + 4$. The eigenvalues are $\lambda = 2, 2$

The matrix $A - 2I = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$. An eigenvector is $\xi = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. $\eta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Then $A\xi = 2\xi$ and $A\eta = \xi + 2\eta$. Take $T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$,

Then $AT = TJ$. where $J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$. $T^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ $h = T^{-1}g = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

$$y' = Jy + h$$

$$\begin{aligned} y_1' &= 2y_1 + y_2 + e^t \\ y_2' &= 2y_2 - e^t \end{aligned}$$

$$\begin{aligned} y_2' - 2y_2 &= -e^t \\ e^{-2t}(y_2' - 2y_2) &= -e^{-t} \\ e^{-2t}(y_2) &= e^{-t} + C_2 \\ y_2 &= e^t + C_2e^{2t} \end{aligned}$$

$$\begin{aligned} y_1' &= 2y_1 + y_2 + e^t \\ y_1' &= 2y_1 + 2e^t + C_2e^{2t} \\ y_1' - 2y_1 &= 2e^t + C_2e^{2t} \\ e^{-2t}(y_1' - 2y_1) &= 2e^{-t} + C_2 \\ e^{-2t}(y_1) &= -2e^{-t} + C_2t + C_1 \\ y_1 &= -2e^t + C_2te^{2t} + C_1e^{2t} \end{aligned}$$

$$\begin{aligned} x_1 &= y_1 = -2e^t + C_2te^{2t} + C_1e^{2t} \\ x_2 &= y_1 + y_2 = -e^t + C_2te^{2t} + (C_1 + C_2)e^{2t} \end{aligned}$$