M309 Quiz # 1a (25 points total) Oct 11, 2006 with **SOLUTIONS**

(1) Find the general solution to $x' = Ax$, where $A = \begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix}$

Solution:

$\det(A - \lambda I) = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4)$. The eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 4$.

$\lambda_1 = 2$, $A - 2I = \begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix}$, $\xi_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$;

$\lambda_2 = 4$, $A - 4I = \begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix}$, $\xi_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

The general solution is $x = C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{4t}$

(2) Solve the initial value problem $x' = Ax$, where $x(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

Solution:

$$\begin{aligned} C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix} &= \begin{bmatrix} 3 \\ 5 \end{bmatrix} \\ -C_1 - 3C_2 &= 3 \\ C_1 + C_2 &= 5 \\ C_1 &= 9 \\ C_2 &= -4 \end{aligned}$$

$$x = 9 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t} - 4 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{4t}$$

M309 Quiz # 1b (25 points total) Oct 11, 2006

(1) Find the general solution to $x' = Ax$, where $A = \begin{bmatrix} 5 & -3 \\ 1 & 1 \end{bmatrix}$

Solution:

$\det(A - \lambda I) = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4)$. The eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 4$.

$\lambda_1 = 2,\ A - 2I = \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix},\quad \xi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix};$

$\lambda_2 = 4,\ A - 4I = \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix},\quad \xi_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

The general solution is $x = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{4t}$

(2) Solve the initial value problem $x' = Ax$, where $x(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

Solution:

$$\begin{aligned} C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} &= \begin{bmatrix} 3 \\ 5 \end{bmatrix} \\ C_1 + 3C_2 &= 3 \\ C_1 + C_2 &= 5 \\ C_1 &= 6 \\ C_2 &= -1 \end{aligned}$$

$x = 6 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{4t}$

M309 Test #1a Oct 20, 2006 Name: ....

(1) Find the general (real) solution to $x' = Ax$, where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix}$

(1b) Find the solution to **(1a)** for which $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(2) Find the general solution to $x' = Ax + g$, where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$, $g = \begin{bmatrix} 3e^t \\ 0 \end{bmatrix}$.

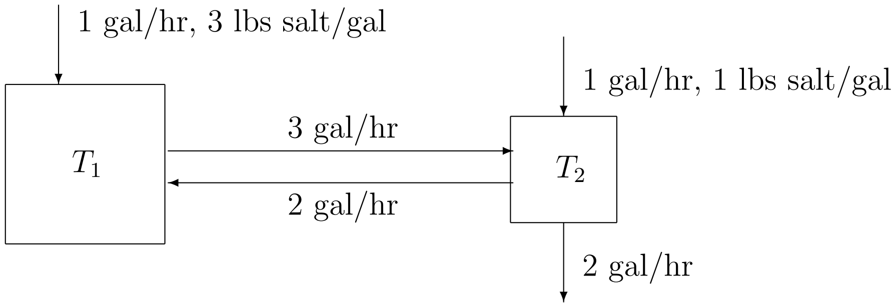
(3a) Do this problem only after you have done the others.

$A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$. Find an invertible matrix T and a matrix J in Jordan form with $AT = TJ$.

(3b) $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$, $g = \begin{bmatrix} e^t \\ 0 \end{bmatrix}$. $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Find the general solution to $x' = Ax + g$.

M309 Quiz # 2a Oct 27, 2000 with **SOLUTIONS**

(1) There are two tanks, with pipes carrying salt water as shown:



T_1 holds 30 gallons of salt water; T_2 holds 10 gallons of salt water.

Let Q_1 be pounds of salt in T_1 at time t;

Let Q_2 be pounds of salt in T_2 at time t;

(1a) Write a system of differential equations for Q_1 and Q_2.

$$\begin{aligned} Q_1' &= -.1Q_1 + .2Q_2 + 3 \\ Q_2' &= .1Q_1 - .4Q_2 + 1 \end{aligned}$$

$$Q_1 = 70; \quad Q_2 = 20$$

Let $R_1 = Q_1 - 70$, $R_2 = Q_2 - 20$. The system is

$$\begin{aligned} R_1' &= -.1R_1 + .2R_2 \\ R_2' &= .1R_1 - .4R_2 \end{aligned}$$

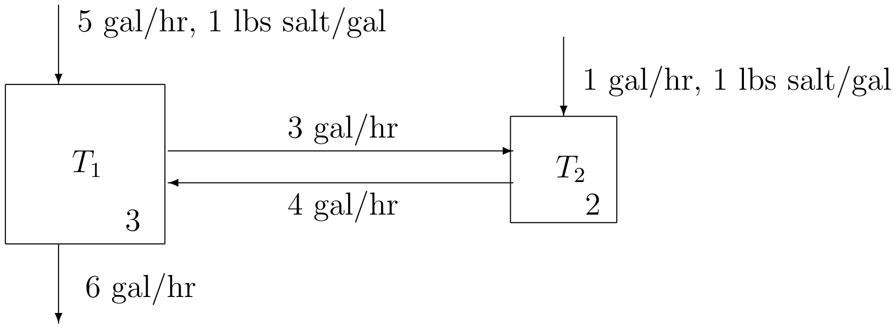
The char poly is $r^2 + 0.5r + 0.02$ The roots are $r = \dfrac{-0.5 \pm \sqrt{0.25 - 0.2}}{2}$

Both roots are negative (because $\sqrt{0.25 - 0.2} < 0.5$), so the point $(70, 20)$ is a <u>stable node</u>.

At $(70, 20)$, the system is in <u>equillibrium</u>

M309 Quiz # 2b Oct 27, 2006, with solutions.

(1) There are two tanks, with pipes carrying salt water as shown:



T_1 holds 3 gallons of salt water; T_2 holds 2 gallon of salt water.

Let Q_1 be pounds of salt in T_1 at time t;

Let Q_2 be pounds of salt in T_2 at time t;

(1a) Write a system of differential equations for Q_1 and Q_2.

SOLUTION

$$\begin{aligned} Q_1' &= -3Q_1 + 2Q_2 + 5 \\ Q_2' &= +Q_1 - 2Q_2 + 1 \end{aligned}$$

(1b) System **(1a)** has one critical point. What is that point; what is type and stability?

SOLUTION $Q_1 = 3; Q_2 = 2;$

Let $R_1 = Q_1 - 3$, $R_2 = Q_2 - 2$. The system is

$$\begin{aligned} R_1' &= -3R_1 + 2R_2 \\ R_2' &= +R_1 - 2R_2 \end{aligned}$$

The char poly is $r^2 + 5r + 6 - 2$. The roots are $r = \dfrac{-5 \pm \sqrt{25-16}}{2}$.

Both roots are negative (because $\sqrt{25-16} < 5$), so the point $(70, 20)$ is a stable node.

(1c) What is the physical interpretation of the values of the critical point?

ANSWER At $(70, 20)$, the system is in equillibrium

M309 Test #2a Nov 8, 2000 Name: ....

(1) Two non-negative populations $x(t)$, $y(t)$ satisfy $\begin{cases} x' = x(4-x-y) \\ y' = y(6-2y-x) \end{cases}$ for $x \geq 0$, $y \geq 0$.

The lines $x+y=4$ and $x+2y=6$ separate the positive quadrant into four regions. Give the signs of the components of direction field in each of the four regions. Sketch a vector of the direction field in each region.

(1b) The point $(2,2)$ is a critical point of **(1a)**. Find the approximating linear system near $(2,2)$. What is the type and stability of $(2,2)$?

(2) Two non-negative populations $x(t)$ and $y(t)$ satisfy $\begin{cases} x' = x(5-y-x) \\ y' = y(-3+x) \end{cases}$,

(2a) The point $(3,2)$ is a critical point. What is its type and stability?

(2b) Plot the direction field at each of the points $(4,2)$, $(3,1)$, $(2,2)$, $(3,3)$,

(3) Non-negative populations $x(t)$ and $y(t)$ satisfy $\begin{cases} x' = x(4-x-2y) \\ y' = y(-3+2x-y) \end{cases}$ for $x \geq 0$, $y \geq 0$.

Find the critical points of this system, and for each critical point give its type and stability

M309 Test #2c Nov 6, 2000 Name: ....

(1) Two non-negative populations $x(t)$, $y(t)$ satisfy $\begin{cases} x' = x(3-x-y) \\ y' = y(4-2y-x) \end{cases}$ for $x \geq 0$, $y \geq 0$.

The lines $x+y=3$ and $x+2y=4$ separate the positive quadrant into four regions. Give the signs of the components of direction field in each of the four regions. Sketch a vector of the direction field in each region.

(1b) The point $(2,1)$ is a critical point of **(1a)** . What is its type and stability?

(2) Two non-negative populations $x(t)$ and $y(t)$ satisfy $\begin{cases} x' = x(-2+y) \\ y' = y(5-y-x) \end{cases}$,

(2a) The point $(3,2)$ is a critical point. What is its type and stability?

(2b) Plot the direction field at each of the points $(4,2)$, $(3,1)$, $(2,2)$, $(3,3)$,

(3) Two non-negative populations $x(t)$ and $y(t)$ satisfy $\begin{cases} x' = x(1-0.5y) \\ y' = y(-.75+0.25x) \end{cases}$

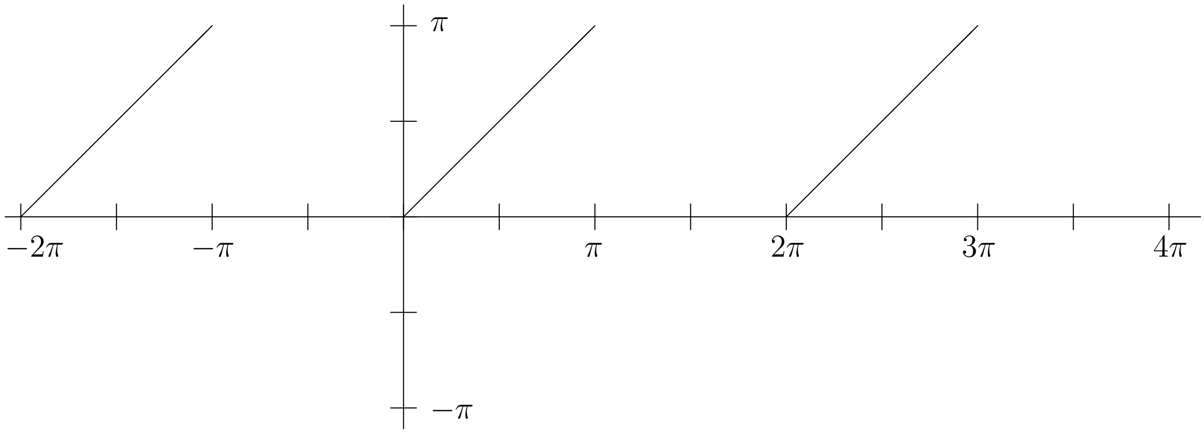
for $x \geq 0$ and $y \geq 0$. Find the critical points of this system, and for each critical point give its type and stability

M309 Quiz # 3 Nov 15, 2006

(1) $f(x) = \left\{ \begin{array}{ll} 0, & \text{for} \quad -\pi \le x < 0 \\ x, & \text{for} \quad 0 \le x < \pi \end{array} \right\}$ $f(x)$ is extended periodically with period 2π.

Sketch the graph of f for $-2\pi \le x \le 4\pi$. Calculate the Fourier Series for $f(x)$.

SOLUTIONS



$$2\pi \frac{a_0}{2} = \int_0^{\pi} x dx = \frac{\pi^2}{2}; \quad \frac{a_0}{2} = \frac{\pi}{4}$$

$$\pi a_m = \int_0^{\pi} x \cos mx dx = \left[\frac{x \sin mx}{m} + \frac{\cos mx}{m^2} \right]_0^{\pi} = \left\{ \begin{array}{l} -\frac{2}{m^2} \text{ if m is odd} \\ 0, \text{ if m is even} \end{array} \right.$$

$$\pi b_m = \int_0^{\pi} x \sin mx dx = \left[-\frac{x \cos mx}{m} + \frac{\sin mx}{m^2} \right]_0^{\pi} = \left\{ \begin{array}{l} -\frac{\pi}{m} \text{ if m is odd} \\ \frac{\pi}{m}, \text{ if m is even} \end{array} \right.$$

$$f(x) = \frac{\pi}{4} + \sum_{k=0}^{\infty} \frac{-2}{\pi(2k+1)^2} \cos(2k+1)x + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \sin mx$$

This problen is very similar to §10.2 #15 in text (assigned for HW VII).

This is also # 7 on the Fourier Series sample problems.

M309 Quiz # 4 Nov 22, 2000 Name ....

(1) The temperature in a rod of length π satisfies the heat equation $u_t = 4u_{xx}$, and the temperature at each end is fixed at 0. The initial temperature is $u(x,0) = \begin{cases} x, & \text{for } 0 \le x < \frac{\pi}{2} \\ \pi - x, & \text{for } \frac{\pi}{2} \le x < \pi \end{cases}$
Find the temperature $u(x,t)$ for all $0 \le x \le \pi$, and all $t \ge 0$.

SOLUTION Let $f(x) = u(x,0)$; $f(x)$ is extended as an <u>odd</u> function for $-\pi \le x \le \pi$. Calculate the Fourier Sine Series for $f(x)$:

$$\pi b_n = 2\int_0^{\pi} f(x)\sin nx dx = 2\Big[\int_0^{\frac{\pi}{2}} x \sin nx dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x)\sin nx dx\Big]$$

For n even, the two integrals are equal, but with opposite signs, $b_{2k} = 0$

For n odd, the two integrals are equal, with same sign. Therefore, for n odd:

$$\pi b_n = 2\int_0^{\pi} f(x)\sin nx dx = 4\int_0^{\frac{\pi}{2}} x \sin nx dx = 4\Big[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\Big]_0^{\frac{\pi}{2}} = 4\Big[\frac{\pm 1}{n^2}\Big]$$

The sign is $+1$ if $n = 1, 5, \ldots$.

The sign is -1 if $n = 3, 7, \ldots$.

Therefore, $\quad f(x) = \dfrac{4}{\pi}\displaystyle\sum_{k=0}^{k=\infty} \frac{(-1)^k}{(2k+1)^2}\sin(2k+1)x$

The solution of the heat equation is $\quad u(x,t) = \dfrac{4}{\pi}\displaystyle\sum_{k=0}^{k=\infty} \frac{(-1)^k}{(2k+1)^2}\sin(2k+1)xe^{-4(2k+1)^2 t}$

M309 Quiz # 4a Nov 27, 2000 Name ....

(1) The temperature in a rod of length π satisfies the heat equation $u_t = 4u_{xx}$, and the temperature at each end is fixed at 0. The initial temperature is $u(x,0) = \begin{cases} x, & \text{for } 0 \le x < \frac{\pi}{2} \\ \pi - x, & \text{for } \frac{\pi}{2} \le x < \pi \end{cases}$
Find the temperature $u(x,t)$ for all $0 \le x \le \pi$, and all $t \ge 0$.

First calculate the Fourier Sine Series for the function $f(x) = u(x,0)$.

$$\pi b_n = 2\int_0^{\pi} f(x)\sin nx dx = 2\Big[\int_0^{\frac{\pi}{2}} x \sin nx dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x)\sin nx dx\Big]$$

For n even, the two integrals are equal, but with opposits signs, $b_{2k} = 0$

For n odd, the two integrals are equal, with same sign. Therefore, for n odd:

$$\pi b_n = 2\int_0^{\pi} f(x)\sin nx dx = 4\int_0^{\frac{\pi}{2}} x\sin nx dx = 4\Big[-\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\Big]_0^{\frac{\pi}{2}} = 4\Big[\frac{\pm 1}{n^2}\Big]$$

The sign is $+1$ if $n = 1, 5, \ldots$.

The sign is -1 if $n = 3, 7, \ldots$.

Therefore, $\quad f(x) = \dfrac{4}{\pi}\displaystyle\sum_{k=0}^{k=\infty} \frac{(-1)^k}{(2k+1)^2}\sin nx$

The solution of the heat eqn is $\quad u(x,t) = \dfrac{4}{\pi}\displaystyle\sum_{k=0}^{k=\infty} \frac{(-1)^k}{(2k+1)^2}\sin nx e^{-4n^2 t}$

M309 Quiz # 5 Dec 1, 2000 Name ....

(1) A metal rod of length π is insulated so that no heat flows into or out of the ends (or sides). The temperature $u(x,t)$ satisfies the heat equation $u_t = u_{xx}$. The initial temperature is $u(x,0) = x$ for $0 \le x < \pi$. Find the temperature $u(x,t)$ for all $0 \le x \le \pi$, and all $t \ge 0$.

SOLUTION Let $f(x) = u(x,0)$; $f(x)$ is extended as an <u>even</u> function for $-\pi \le x \le \pi$.

Calculate the Fourier Cosine Series for $f(x)$:

$$2\pi \frac{a_0}{2} = 2\int_0^\pi x dx = 2\frac{\pi^2}{2} = \pi^2 \qquad \frac{a_0}{2} = \frac{\pi}{2}$$

$$\pi a_n = 2\int_0^\pi f(x)\cos nx dx \quad = 2\int_0^\pi x\cos nx dx$$

$$= 2\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^\pi = \frac{2}{n^2}\left[\cos n\pi - 1\right]$$

$$a_n = -\frac{4}{\pi n^2} \quad \text{if } n \text{ is odd}$$

$$a_n = 0 \qquad \text{if } n \text{ is even}$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{k=\infty} \frac{1}{(2k+1)^2}\cos\Big((2k+1)x\Big)$$

The solution of the heat equation is: $u(x,t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{k=\infty} \frac{1}{(2k+1)^2}\cos\Big((2k+1)x\Big)e^{-(2k+1)^2 t}$

M309 Quiz # 5a Dec 5, 2000 Name ....

(1) The temperature $u(x,t)$ in a rod of length π satisfies the heat equation $u_t = 4u_{xx}$. The rod is insulated so that no heat flows into or out of the ends. The initial temperature is $u(x,0) = \pi - x$ for $0 \le x < \pi$. Find the temperature $u(x,t)$ for all $0 \le x \le \pi$, and all $t \ge 0$.

SOLUTION Let $f(x) = u(x,0)$; $f(x)$ is extended as an even function for $-\pi \le x \le \pi$.

Calculate the Fourier Cosine Series for $f(x)$:

$$2\pi\frac{a_0}{2} = 2\int_0^\pi x dx = 2\frac{\pi^2}{2} = \pi^2 \qquad \frac{a_0}{2} = \frac{\pi}{2}$$

$$\pi a_n = 2\int_0^\pi f(x)\cos nx dx \quad = 2\int_0^\pi x\cos nx dx$$

$$= 2\Big[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\Big]_0^\pi = \frac{2}{n^2}\Big[\cos n\pi - 1\Big]$$

$$a_n = -\frac{4}{\pi n^2} \quad \text{if } n \text{ is odd}$$

$$a_n = 0 \quad \text{if } n \text{ is even}$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{k=\infty}\frac{1}{(2k+1)^2}\cos\Big((2k+1)x\Big)$$

The solution of the heat equation is: $$u(x,t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{k=\infty}\frac{1}{(2k+1)^2}\cos\Big((2k+1)x\Big)e^{-(2k+1)^2 t}$$

M309 Test #3a Dec 13, 2006 with **SOLUTIONS**

(1a) Let $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$ Find the solution to $x' = Ax$, with $x(0) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

Solution: $\det \begin{bmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{bmatrix} = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$

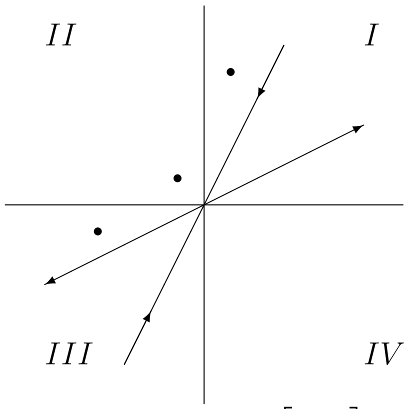
For the eigenvalue $\lambda_1 = -1$, the eigenvector is $\xi_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

For the eigenvalue $\lambda_2 = +2$, the eigenvector is $\lambda_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

The general solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}$

If $x(0) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, then $C_1 = 3, \quad C_2 = -1$. So $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}$

(1b) The trajectory starts in I, then goes to II, and finally goes into III, where it remains.



(1c) When $t = 1000$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2e^{2000} + 3e^{-1000} \\ -e^{2000} + 6e^{-1000} \end{bmatrix}$ which is in III

(2) The temperature in rod of length π satisfies the heat equation $u_t = 4u_{xx}$. The temperature at each end is fixed at 0, and its initial temperature is $u(x,0) = \sin x + \sin 3x$ for $0 \le x \le \pi$. Find the temperature $u(x,t)$ for all $0 \le x \le \pi$, and all $t \ge 0$.

Solution: $u(x,t) = \sin x \cdot e^{-4 \cdot t} + \sin 3x \cdot e^{-36 \cdot t}$

(3) $u(x,y)$ satisfies Laplace's Equation $u_{xx} + u_{yy} = 0$ in the square $0 \le x \le \pi$, $0 \le y \le \pi$, and $u(x,y)$ satisfies the boundary conditions:

$$\begin{cases} u(0,y) = 0 & \text{for } 0 \le y \le \pi; \\ u(\pi,y) = \sin 2y & \text{for } 0 \le y \le \pi \\ u(x,0) = 0, & \text{for } 0 \le x \le \pi \\ u(x,\pi) = 0 & \text{for } 0 \le x \le \pi. \end{cases}$$

Find the solution $u(x,y)$ for all (x,y) in the square.

Solution: $u(x,y) = \dfrac{1}{\sinh 2\pi} \sinh 2x \cdot \sin 2y$

(4a) $f(x) = x$, for $0 \le x \le \pi$. Extend $f(x)$ as an <u>odd</u> function of period 2π. Calculate the Fourier Series for $f(x)$.

Solution: This is a Fourier Sine Series, so all $a_n = 0$. For $n \ge 1$,

$$\pi b_n = 2\int_0^\pi x \sin nx dx = \left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^\pi = \begin{cases} +\frac{2\pi}{n} & \text{if n is odd} \\ -\frac{2\pi}{n}, & \text{if n is even} \end{cases}$$

$$b_n = \frac{2}{n} \cdot (-1)^{n+1}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n} \cdot (-1)^{n+1} \sin nx$$

(4b) Solution: Parseval's equality for $f(x)$ is $\int_{-\pi}^{\pi} f(x)^2 dx = \pi\Big(\sum_{n=1}^{n=\infty} b_n^2\Big)$

$$\frac{2}{3}\pi^3 = \pi\Big(\sum_{n=1}^{n=\infty} \frac{4}{n^2}\Big)$$

$$\frac{\pi^2}{6} = \sum_{n=0}^{n=\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots$$

(5) The populations of two species satisfy: $\begin{cases} x' = x(1.125 - x - 0.5y) \\ y' = y(-1 + x) \end{cases}$

(5a) Find all critical points of the system.

(5b) For each critical point, find the eigenvalues of the approximate linear system.

(5c) Classify each critical point as to type and stability.

Solution: The critical points are $(0,0)$, $(1.125, 0)$, $(1, 0.5)$.

(1) At $(0,0)$, the approx. linear system is $\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 1.125 & + & 0 \\ 0 & - & v \end{bmatrix}$,

The eigenvalues are 1.125, and -1, one positive, one negative, so $(0,0)$, is a saddle point (unstable).

(2) At $(1.125, 0)$, the approx. linear system is $\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -1.125u & - & .5625v \\ 0 & + & .125v \end{bmatrix}$

The eigenvalues are -1.125, and $+0.125$, one positive, one negative, so $(1.125, 0)$ is a saddle point (unstable).

(3) At $(1, 0.25)$, the approx linear system is $\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -u & - & 0.5v \\ .25u & + & 0 \end{bmatrix}$

The eigenvalues are $\dfrac{-1 \pm \sqrt{0.5}}{2}$, both negative, so $(1, 0.25)$,is a stable node.

(6) The position of a string of length π satisfies the wave equation $u_{tt} = 9u_{xx}$. The string is initially motionless, the ends are fixed at 0, and its initial position is

$u(x,0) = \begin{cases} x, & \text{for } 0 \le x < \frac{\pi}{2} \\ \pi - x, & \text{for } \frac{\pi}{2} \le x < \pi \end{cases}$ Find the position $u(x,t)$ for all $0 \le x \le \pi$, and $t \ge 0$.

Solution: First find the Fourier Sine Series for $f(x) = u(x,0)$.

$$\pi b_n = 2\int_0^\pi f(x)\sin nx dx = 2\Big(\int_0^{\frac{\pi}{2}} x\sin nx dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x)\sin n)x dx\Big)$$

For n even, the two integrals are equal, but with opposite signs, $b_{2k} = 0$

For n odd, the two integrals are equal, with same sign. Therefore, for n odd:

$$\pi b_n = 2\int_0^\pi f(x)\sin nx dx = 4\int_0^{\frac{\pi}{2}} x\sin nx dx = 4\Big[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\Big]_0^{\frac{\pi}{2}} = 4\Big[\frac{\pm 1}{n^2}\Big]$$

The sign is $+1$ if $n = 1, 5, \ldots$.

The sign is -1 if $n = 3, 7, \ldots$.

Therefore, $\quad f(x) = \dfrac{4}{\pi}\displaystyle\sum_{k=0}^{k=\infty} \frac{(-1)^k}{(2k+1)^2}\sin(2k+1)x$

The solution of the wave equation is $\quad u(x,t) = \dfrac{4}{\pi}\displaystyle\sum_{k=0}^{k=\infty} \frac{(-1)^k}{(2k+1)^2}\sin(2k+1)x\cos((2k+1)\cdot 3\cdot t)$

M309 Test #3c Dec 11, 2006 with **SOLUTIONS**

(1a) Let $A = \begin{bmatrix} 3 & 2 \\ -2 & -2 \end{bmatrix}$ Find the solution to $x' = Ax$, with $x(0) = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$.

Solution: $\det \begin{bmatrix} 3-\lambda & 2 \\ -2 & -2-\lambda \end{bmatrix} = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$

For the eigenvalue $\lambda_1 = -1$, the eigenvector is $\xi_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

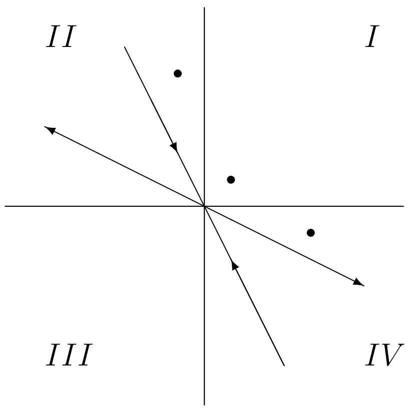
For the eigenvalue $\lambda_2 = +2$, the eigenvector is $\lambda_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

The general solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{2t}$

If $x(0) = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$, then $C_1 = -3$, $C_2 = 1$. So $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (-3) \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{2t}$

(1b) The trajectory starts in II, then to I, and finally into IV

(1c) When $t = 1000$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2e^{2000} - 3e^{-1000} \\ -e^{2000} + 6e^{-1000} \end{bmatrix}$ which is in IV



(2) A string length π satisfies the wave equation $u_{tt} = 4u_{xx}$. The string is initially motionless, the ends are fixed at 0, and its initial position is $u(x,0) = \sin 3x + \sin 5x$ for $0 \le x \le \pi$. Find the position $u(x,t)$ for all $0 \le x \le \pi$, and all $t \ge 0$.

Solution: $u(x,t) = \sin 3x \cdot \cos 6t + \sin 5x \cdot \cos 10t$

(3) Find the solution of Laplace's Equation $u_{xx} + u_{yy} = 0$ in the rectangle $0 \le x \le \pi$, $0 \le y \le \pi$, satisfying the boundary conditions: $u(0,y) = 0$, for $0 < y < \pi$; $u(\pi, y) = 0$ for $0 < y < \pi$; $u(x,0) = 0$, $0 < x < \pi$; and $u(x,\pi) = \sin 2x$ for $0 < x < \pi$.

Solution: $u(x,y) = \dfrac{1}{\sinh 2\pi} \sin 2x \cdot \sinh 2y$

(4a) $f(x) = x$, for $0 \le x \le \pi$. Extend $f(x)$ as an <u>even</u> function of period 2π. Calculate the Fourier Series for $f(x)$.

$f(x) = \dfrac{a_0}{2} + \sum_{m=1}^{m=\infty} a_m \cos mx$, a Fourier Cosine Series ($f(x)$ is even, so all $b_n = 0$).

$$\frac{a_0}{2} = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^\pi x \cos nx dx = \frac{2}{\pi}\left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^\pi = \frac{-4}{\pi n^2} \text{ if } n \text{ is odd, 0 if } n \text{ is even.}$$

$$f(x) = \frac{\pi}{2} + \sum_{k=0}^{k=\infty} \frac{-4}{\pi(2k+1)^2} \cos(2k+1)x$$

(4b) Parseval's equality for $f(x)$ is $\int_{-\pi}^{\pi} f(x)^2 dx = \pi\left(\dfrac{a_0^2}{2} + \sum_{n=0}^{n=\infty} a_n^2\right)$

$$\frac{2}{3}\pi^3 = \pi\left(\frac{\pi^2}{2} + \sum_{k=0}^{k=\infty} \frac{16}{\pi^2(2k+1)^4}\right) = \frac{\pi^3}{2} + \sum_{k=0}^{k=\infty} \frac{16}{\pi(2k+1)^4}$$

$$\frac{\pi^3}{6} = \sum_{k=0}^{k=\infty} \frac{16}{\pi(2k+1)^4}, \quad \text{So}$$

$$\frac{\pi^4}{96} = \sum_{k=0}^{k=\infty} \frac{1}{(2k+1)^4} = 1 + \frac{1}{81} + \frac{1}{625} + \dots$$

(5) The populations of two species satisfy: $x' = x(1 - 0.5x - 0.5y)$; $y' = y(-0.25 + 0.5x)$.

(5a) Find all critical points of the system.

(5b) For each critical point, find the eigenvalues and eigenvectors of the approximate linear system.

(5c) Classify each critical point as to type and stability.

Solutions: The critical points are $A = (0,0), B = (2,0), C = (0.5, 1.5)$.

(A) At $(0,0)$, the approximating linear system is $\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} u & + & 0 \\ 0 & - & 0.25v \end{bmatrix}$

The eigenvalues are 1, and $-.25$, one positive, one negative, so $(0,0)$ is a saddle point (unstable).

(B) At $(2,0)$, the approximating linear system is $\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -u & - & v \\ 0 & + & 0.75v \end{bmatrix}$

The eigenvalues are -1, and $+0.75$, one positive, one negative, so $(2,0)$ is a saddle point (unstable).

(C) At $(0.5, 1.5)$, the approximating linear system is $\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -.25u & - & .25v \\ .75u & + & 0 \end{bmatrix}$

The eigenvalues are $\dfrac{-.25 \pm \sqrt{-.5625}}{2}$, which are complex, with negative real part, so $(0.5, 1.5)$ is a spiral inwards point (stable).

(6) The temperature in a rod of length π satisfies the heat equation $u_t = 4u_{xx}$, and the temperature at each end is fixed at 0. The initial temperature is $u(x,0) = \begin{cases} x, & \text{for } 0 \le x < \frac{\pi}{2} \\ \pi - x, & \text{for } \frac{\pi}{2} \le x < \pi \end{cases}$
Find the temperature $u(x,t)$ for all $0 \le x \le \pi$, and all $t \ge 0$.

Solution: First find the Fourier Sine Series for $f(x) = u(x,0)$.

$$\pi b_n = 2\int_0^{\pi} f(x)\sin nx dx = 2\Big(\int_0^{\frac{\pi}{2}} x \sin nx dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x)\sin n)x dx\Big)$$

For n even, the two integrals are equal, but with opposite signs, therefore $b_{2k} = 0$

For n odd, the two integrals are equal, with same sign. Therefore, for n odd:

$$\pi b_n = 2\int_0^{\pi} f(x)\sin nx dx = 4\int_0^{\frac{\pi}{2}} x \sin nx dx = 4\Big[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\Big]_0^{\frac{\pi}{2}} = 4\Big[\frac{\pm 1}{n^2}\Big]$$

The sign is $+1$ if $n = 1, 5, \ldots$.
The sign is -1 if $n = 3, 7, \ldots$.

Therefore, $f(x) = \dfrac{4}{\pi}\displaystyle\sum_{k=0}^{k=\infty} \frac{(-1)^k}{(2k+1)^2}\sin(2k+1)x$

The solution of the heat equation is $u(x,t) = \dfrac{4}{\pi}\displaystyle\sum_{k=0}^{k=\infty} \frac{(-1)^k}{(2k+1)^2}\sin(2k+1)x e^{-4(2k+1)^2 t}$