

M309 Test #3a Dec 13, 2006 with **Solutions**

(1a) Let $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$ Find the solution to $x' = Ax$, with $x(0) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

Solution: $\det \begin{bmatrix} 3 - \lambda & -2 \\ 2 & -2 - \lambda \end{bmatrix} = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$

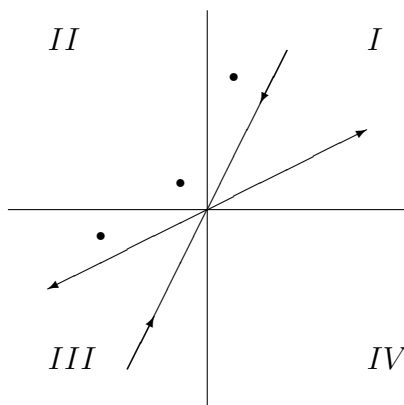
For the eigenvalue $\lambda_1 = -1$, the eigenvector is $\xi_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

For the eigenvalue $\lambda_2 = +2$, the eigenvector is $\xi_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

The general solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}$

If $x(0) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, then $C_1 = 3$, $C_2 = -1$. So $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}$

(1b) The trajectory starts in I, then goes to II, and finally goes into III, where it remains.



(1c) When $t = 1000$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2e^{2000} + 3e^{-1000} \\ -e^{2000} + 6e^{-1000} \end{bmatrix}$ which is in III

(2) The temperature in rod of length π satisfies the heat equation $u_t = 4u_{xx}$. The temperature at each end is fixed at 0, and its initial temperature is $u(x, 0) = \sin x + \sin 3x$ for $0 \leq x \leq \pi$. Find the temperature $u(x, t)$ for all $0 \leq x \leq \pi$, and all $t \geq 0$.

Solution: $u(x, t) = \sin x \cdot e^{-4t} + \sin 3x \cdot e^{-36t}$

(3) $u(x, y)$ satisfies Laplace's Equation $u_{xx} + u_{yy} = 0$ in the square $0 \leq x \leq \pi$, $0 \leq y \leq \pi$, and $u(x, y)$ satisfies the boundary conditions:

$$\begin{cases} u(0, y) = 0 & \text{for } 0 \leq y \leq \pi; \\ u(\pi, y) = \sin 2y & \text{for } 0 \leq y \leq \pi \\ u(x, 0) = 0, & \text{for } 0 \leq x \leq \pi \\ u(x, \pi) = 0 & \text{for } 0 \leq x \leq \pi. \end{cases} \quad \text{Find the solution } u(x, y) \text{ for all } (x, y) \text{ in the square.}$$

Solution: $u(x, y) = \frac{1}{\sinh 2\pi} \sinh 2x \cdot \sin 2y$

(4a) $f(x) = x$, for $0 \leq x \leq \pi$. Extend $f(x)$ as an odd function of period 2π . Calculate the Fourier Series for $f(x)$.

Solution: This is a Fourier Sine Series, so all $a_n = 0$. For $n \geq 1$,

$$\pi b_n = 2 \int_0^\pi x \sin nx dx = \left[-\frac{x \cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^\pi = \begin{cases} +\frac{2\pi}{n} & \text{if } n \text{ is odd} \\ -\frac{2\pi}{n}, & \text{if } n \text{ is even} \end{cases}$$

$$b_n = \frac{2}{n} \cdot (-1)^{n+1}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n} \cdot (-1)^{n+1} \sin nx$$

(4b) **Solution:** Parseval's equality for $f(x)$ is $\int_{-\pi}^{\pi} f(x)^2 dx = \pi \left(\sum_{n=1}^{n=\infty} b_n^2 \right)$

$$\frac{2}{3} \pi^3 = \pi \left(\sum_{n=1}^{n=\infty} \frac{4}{n^2} \right)$$

$$\frac{\pi^2}{6} = \sum_{n=0}^{n=\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots$$

(5) The populations of two species satisfy: $\begin{cases} x' = x(1.125 - x - 0.5y) \\ y' = y(-1 + x) \end{cases}$

(5a) Find all critical points of the system.

(5b) For each critical point, find the eigenvalues of the approximate linear system.

(5c) Classify each critical point as to type and stability.

Solution: The critical points are $A = (0, 0)$, $B = (1.125, 0)$, $C = (1, 0.5)$.

(A) At $(0, 0)$, the approx. linear system is $\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 1.125 & + & 0 \\ 0 & - & v \end{bmatrix}$,

The eigenvalues are 1.125, and -1 , one positive, one negative, so $(0, 0)$, is a saddle point (unstable).

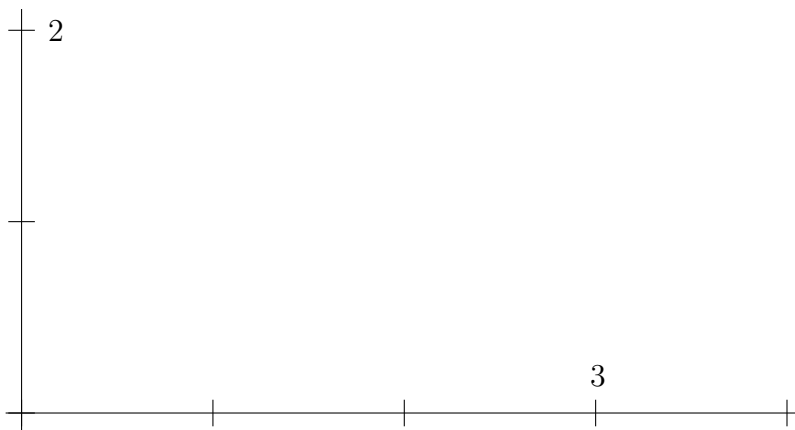
(B) At $(1.125, 0)$, the approx. linear system is $\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -1.125u & - & .5625v \\ 0 & + & .125v \end{bmatrix}$

The eigenvalues are -1.125 , and $+0.125$, one positive, one negative, so $(1.125, 0)$ is a saddle point (unstable).

(C) At $(1, 0.25)$, the approx linear system is $\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -u & - & 0.5v \\ .25u & + & 0 \end{bmatrix}$

The eigenvalues are $\frac{-1 \pm \sqrt{0.5}}{2}$, both negative, so $(1, 0.25)$, is a stable node.

(5d) Plot enough of the direction field to determine the behavior of the solution $(x(t), y(t))$ as $t \rightarrow \infty$ of the trajectory which starts at $(2, 1)$.



(6) The position of a string of length π satisfies the wave equation $u_{tt} = 9u_{xx}$. The string is initially motionless, the ends are fixed at 0, and its initial position is

$$u(x, 0) = \begin{cases} x, & \text{for } 0 \leq x < \frac{\pi}{2} \\ \pi - x, & \text{for } \frac{\pi}{2} \leq x < \pi \end{cases} \quad \text{Find the position } u(x, t) \text{ for all } 0 \leq x \leq \pi, \text{ and } t \geq 0.$$

Solution: First find the Fourier Sine Series for $f(x) = u(x, 0)$.

$$\pi b_n = 2 \int_0^\pi f(x) \sin nx dx = 2 \left(\int_0^{\frac{\pi}{2}} x \sin nx dx + \int_{\frac{\pi}{2}}^\pi (\pi - x) \sin nx dx \right)$$

For n even, the two integrals are equal, but with opposite signs, $b_{2k} = 0$

For n odd, the two integrals are equal, with same sign. Therefore, for n odd:

$$\pi b_n = 2 \int_0^\pi f(x) \sin nx dx = 4 \int_0^{\frac{\pi}{2}} x \sin nx dx = 4 \left[-\frac{x \cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^{\frac{\pi}{2}} = 4 \left[\frac{\pm 1}{n^2} \right]$$

The sign is +1 if $n = 1, 5, \dots$

The sign is -1 if $n = 3, 7, \dots$

Therefore,
$$f(x) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sin(2k+1)x$$

The solution of the wave equation is
$$u(x, t) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sin(2k+1)x \cos((2k+1) \cdot 3 \cdot t)$$