Math 424B

Autumn 2004

Solutions to Homework 5

3.30. Let (M, d) be a metric space. Since the union of a finite collection of closed sets is closed, it suffices to show that the one point set $\{x\}$ is always closed. We will show that $M - \{x\}$ is open. Indeed, suppose $y \neq x$. Then let $\epsilon = d(x, y) > 0$, and observe that if $z \in B_M(y, \epsilon)$, then $z \neq x$. Hence $B_M(y, \epsilon) \subset M - \{x\}$, so that y is in the interior of $M - \{x\}$. Since y was arbitrary, we have that $M - \{x\}$ is open.

3.31. a. We prove that $M - \bar{B}(a, r)$ is open. Let $y \in M - \bar{B}(a, r)$, so that $d(y, a) > r$. Then let $\epsilon = d(y, a) - r$. If $z \in B(y, \epsilon)$, then

$$d(a, z) \geq d(a, y) - d(z, y) > d(a, y) - \epsilon = r,$$

so that $z \in M - \bar{B}(a, r)$. This shows that $B(y, \epsilon) \subset M - \bar{B}(a, r)$ and hence that $M - \bar{B}(a, r)$ is open.

b. Let M be any set with more than one element and let d be the discrete metric. Then $B(a, 1) = \{a\}$, so the closure of $B(a, 1)$ is $\{a\}$. However, $\bar{B}(a, 1) = M$.

3.32. We have $A \subset S \subset \bar{A}$ and $S \subset T \subset \bar{S}$. Hence $A \subset T$, and we need only show that $T \subset \bar{A}$. But $S \subset \bar{A}$ implies that $\bar{S}$ is a subset of the closure of $\bar{A}$, which is $\bar{A}$ itself. Thus $T \subset \bar{S} \subset \bar{A}$, completing the proof.

3.38. First suppose S is compact in (T, d), and let F be an open cover of S in (M, d). Then $S \subset \bigcup_{A \in F} A$, and, since $S \subset T$, we thus have $S \subset \bigcup_{A \in F} (A \cap T)$. Therefore, $F' = \{\, A \cap T : A \in F \,\}$ is an open cover of S in T, so, by compactness, there exists a finite subcover $\{A_1 \cap T, \ldots, A_k \cap T\}$. Since

$$S \subset (A_1 \cap T) \cup \cdots \cup (A_k \cap T) \subset A_1 \cup \cdots \cup A_k,$$

it follows that $\{A_1, \ldots, A_k\} \subset F$ is a finite cover of S. F was arbitrary, so S is compact in (M, d).

Conversely, suppose S is compact in (M, d), and let F be an open cover of S in (T, d). For each $A \in F$, we may choose an open set B_A in (M, d) such that $A = T \cap B_A$. Then let $F' = \{\, B_A : A \in F \,\}$. F' is an open cover of S in M, so, by compactness, there is a finite subcover $\{B_{A_1}, \ldots, B_{A_k}\}$. Hence

$$S = S \cap T \subset (B_{A_1} \cup \cdots \cup B_{A_k}) \cap T = A_1 \cup \cdots \cup A_k.$$

Therefore, $\{A_1, \ldots, A_k\} \subset F$ is a cover of S, and S is compact in (T, d).

3.42. S is certainly bounded, and S is closed, because $S = [a, b] \cap \mathbb{Q}$ and $[a, b]$ is closed in $\mathbb{R}$. But S is not compact in $\mathbb{R}$ because it is not closed there—its closure is $[a, b]$. By 3.38, S cannot be compact in $\mathbb{Q}$.

3.43. $M - (\overline{M - A}) \subset M - (M - A) = A$. Since $M - (\overline{M - A})$ is open, every point $x \in M - (\overline{M - A})$ lies in a ball $B_M(x, \epsilon)$ contained in $M - (\overline{M - A})$ and thus lies in A. This shows that $x \in A^0$, so that $M - (\overline{M - A}) \subset A^0$. On the other hand, $M - A^0$ is a closed set containing $M - A$. But $\overline{M - A}$ is the smallest closed set containing $M - A$, so $\overline{M - A} \subset M - A^0$. Hence $A^0 = M - (M - A^0) \subset M - (\overline{M - A})$, and therefore, $A^0 = M - (\overline{M - A})$.

3.50. Let $M = \mathbb{R}$ with the usual metric, and let $A = \mathbb{Q}$, $B = \mathbb{R} - \mathbb{Q}$.