Exercises from 1.1 of Patty:

3.
\[ \|x + y\|^2 = (x + y) \cdot (x + y) = \|x\|^2 + 2x \cdot y + \|y\|^2 \]
\[ \leq \|x\|^2 + 2|x| \cdot |y| + \|y\|^2\]
\[ \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \quad \text{(by Cauchy-Schwartz)} \]
\[ = (\|x\| + \|y\|)^2. \]

This proves that \( \|x + y\| \leq \|x\| + \|y\| \).

12. Clearly \( d((x_1, y_1), (x_2, y_2)) = d((x_2, y_2), (x_1, y_1)) \) and \( d((x_1, y_1), (x_1, y_1)) = 0 \).
If \( d((x_1, y_1), (x_2, y_2)) = 0 \), then \( \max\{d_1(x_1, x_2), d_2(y_1, y_2)\} = 0 \). Since \( d_1(x_1, x_2) \) and \( d_2(y_1, y_2) \) are both nonnegative, we must have \( d_1(x_1, x_2) = d_2(y_1, y_2) \).
But this implies that \( x_1 = x_2 \) and \( y_1 = y_2 \). Finally, we verify the triangle inequality. Indeed,
\[ d_1(x_1, x_3) \leq d_1(x_1, x_2) + d_1(x_2, x_3) \leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)) \]
\[ d_2(y_1, y_3) \leq d_2(y_1, y_2) + d_2(y_2, y_3) \leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)), \]

so
\[ d((x_1, y_1), (x_3, y_3)) = \max\{d_1(x_1, x_3), d_2(y_1, y_3)\} \leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)). \]

13. \( d \) is not a metric on \( X \times Y \) if \( Y \) (or \( X \)) has at least two elements and the other set is nonempty. For then \( d((x, y_1), (x, y_2)) = 0 \) but \( (x, y_1) \neq (x, y_2) \) for \( y_1 \neq y_2 \).

19. Let \( x_0 \in X \) and \( \varepsilon > 0 \). Since
\[ d(x_0, a) \leq d(x_0, x) + d(x, a) \]
for all \( x \in X \), \( a \in A \), it follows that
\[ f(x_0) = \inf\{ d(x_0, a) : a \in A \} \leq d(x_0, x) + \inf\{ d(x, a) : a \in A \} \]
\[ = d(x_0, x) + f(x). \]

Interchanging \( x \) and \( x_0 \) yields
\[ f(x) \leq d(x, x_0) + f(x_0); \]

hence
\[ |f(x) - f(x_0)| \leq d(x, x_0). \]

Therefore, \( d(x, x_0) < \varepsilon \Rightarrow |f(x) - f(x_0)| < \varepsilon \), so \( f \) is continuous.
22. Let \( a \in X_1 \) and \( \epsilon > 0 \). We need to find a \( \delta > 0 \) such that
\[
d_1(x_1, a) < \delta \Rightarrow d_3((g \circ f)(x_1), (g \circ f)(a)) < \epsilon.
\]
g is continuous at \( f(a) \), so we may choose \( \delta_1 > 0 \) such that
\[
d_2(x_2, f(a)) < \delta_1 \Rightarrow d_3(g(x_2), g(f(a))) < \epsilon.
\]
Since \( f \) is continuous at \( a \), there exists \( \delta > 0 \) such that
\[
d_1(x_1, a) < \delta \Rightarrow d_2(f(x_1), f(a)) < \delta_1.
\]
But from above,
\[
d_2(f(x_1), f(a)) < \delta_1 \Rightarrow d_3(g(f(x_1)), g(f(a))) < \epsilon,
\]
so our chosen \( \delta \) satisfies the desired implication.

Extra Exercise 1: Let \( T_\rho \) be the topology induced by \( \rho \) and \( T_d \) the topology induced by \( d \). We will show that the inequality
\[
ad(x, y) \leq \rho(x, y)
\]
implies that \( T_\rho \) is finer than \( T_d \). Since we also have
\[
b^{-1} \rho(x, y) \leq d(x, y),
\]
we then have that \( T_d \) is finer than \( T_\rho \); hence \( T_d = T_\rho \).

Suppose then that \( U \in T_d \) and let \( x_0 \in U \). Choose \( \epsilon > 0 \) such that \( B_d(x_0, \epsilon) \subset U \).
If \( \rho(x, x_0) < \epsilon a \), then \( d(x, x_0) < \epsilon \), so \( x \in U \). This proves that \( B_\rho(x, \epsilon a) \subset U \), and hence, since \( x_0 \in U \) was arbitrary, that \( U \) is in \( T_\rho \).

Now let \( d \) denote the standard metric and let \( \rho \) denote the square metric on \( \mathbb{R}^n \).
If \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \), then
\[
|x_i - y_i| = \sqrt{|x_i - y_i|^2} \leq \|x - y\|
\]
and thus \( \rho(x, y) \leq d(x, y) \). On the other hand,
\[
d(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}
\]
and \( (x_i - y_i)^2 \leq \rho(x, y)^2 \) for all \( i \); hence
\[
d(x, y) \leq \sqrt{n\rho(x, y)^2} = \sqrt{n} \rho(x, y).
\]
By what we did above, these two inequalities imply that \( d \) and \( \rho \) induce the same topology.

Extra Exercise 2: Suppose the topology on \( X \) is induced by a metric \( d \), and that \( x_1, x_2 \in X \) with \( x_1 \neq x_2 \). Then let \( r = d(x_1, x_2) > 0 \). \( B_d(x_1, r) \) is an open set which contains \( x_1 \) but not \( x_2 \); thus there is an open set which is neither the empty set nor is all of \( X \). Therefore, \( X \) does not have the trivial topology.