Exercises from Munkres:

§13.1. We show that \( \bigcup_{U \in A} U = A \). Since the union of a collection of open sets is open, this proves that \( A \) is open. To prove that \( \bigcup_{U \in A} U = A \), first observe that \( \bigcup_{U \in A} U \subset A \), since each \( U \) in the union is a subset of \( A \). Now suppose \( x \in A \). Then, by assumption, there exists \( U \subset A \) with \( x \in U \). Hence \( x \in \bigcup_{U \in A} U \), so, since \( x \) was arbitrary, \( A \subset \bigcup_{U \in A} U \). Therefore, \( A = \bigcup_{U \in A} U \), completing the proof.

§13.3. First observe that \( \emptyset \in \mathcal{T}_c \), since \( X - \emptyset = X \), and that \( X \in \mathcal{T}_c \), since \( X - X = \emptyset \) and \( \emptyset \) is countable. Next suppose that \( \{U_\alpha\} \subset \mathcal{T}_c \). If every \( U_\alpha \neq \emptyset \), then \( \bigcup_\alpha U_\alpha = \emptyset \in \mathcal{T}_c \). If some \( U_\beta \neq \emptyset \), then \( X - U_\beta \) is countable and

\[
X - \bigcup_\alpha U_\alpha = \bigcap_\alpha (X - U_\alpha) \subset X - U_\beta.
\]

Hence \( X - \bigcup_\alpha U_\alpha \) is countable, so \( \bigcup_\alpha U_\alpha \in \mathcal{T}_c \). Finally, suppose that \( U_1 \) and \( U_2 \) are in \( \mathcal{T}_c \). If either \( U_1 \) or \( U_2 \) is empty, then \( U_1 \cap U_2 = \emptyset \in \mathcal{T}_c \). Otherwise, \( X - U_1 \) and \( X - U_2 \) are both countable, so

\[
X - (U_1 \cap U_2) = (X - U_1) \cup (X - U_2)
\]

is countable and thus \( U_1 \cap U_2 \in \mathcal{T}_c \). These observations imply that \( \mathcal{T}_c \) is a topology on \( X \).

However, \( \mathcal{T}_\infty \) is not necessarily a topology on \( X \). For example, let \( X = \mathbb{N} \), and let \( U \) be the set of even numbers and \( V \) be the set of odd numbers greater than 1. Then \( U \) and \( V \) are both in \( \mathcal{T}_\infty \), but \( X - (U \cup V) = \{1\} \), so \( U \cup V \notin \mathcal{T}_\infty \). In fact, \( \mathcal{T}_\infty \) is never a topology if \( X \) is infinite.

§13.8. a. Each element of \( \mathcal{B} \) is open in the standard topology so it suffices to show that if \( x \in U \) and \( U \) is open in the standard topology, then there exists \( B \in \mathcal{B} \) with \( x \in B \subset U \). But, since \( U \) is open, there exists \( \epsilon > 0 \) such that \( (x - \epsilon, x + \epsilon) \subset U \). Now choose \( a \in \mathbb{Q} \) with \( x - \epsilon < a < x \) and \( b \in \mathbb{Q} \) with \( x < b < x + \epsilon \); then \( x \in (a, b) \subset U \) and \( (a, b) \in \mathcal{B} \).

b. That \( \mathcal{C} \) generates a topology \( \mathcal{T} \) follows from the same argument that we used to construct the lower limit topology \( \mathcal{T}' \) on \( \mathbb{R} \). Clearly \( \mathcal{T} \subset \mathcal{T}' \). However, I claim that if \( u \) is irrational and \( u < v \), then \( [u, v) \notin \mathcal{T} \). Since \( [u, v) \in \mathcal{T}' \), this proves that \( \mathcal{T} \neq \mathcal{T}' \). To prove the claim, observe that if \( [u, v) \in \mathcal{T} \), then it would have to be the union of elements of \( \mathcal{C} \). In particular, there would have to be \( a, b \in \mathbb{Q} \) such that \( u \in [a, b) \subset [u, v) \). But \( u \) is irrational, so \( u \in [a, b) \Rightarrow u > a \), whence \( [a, b) \notin [u, v) \). Therefore, \( [u, v) \notin \mathcal{T} \).
§16.3. \( A = (-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1) \) is open in \( \mathbb{R} \) and hence open in \( Y \).
\( B = [-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1] \) is not open in \( \mathbb{R} \) but is open in \( Y \) since \( B = Y \cap (-\infty, -\frac{1}{2}) \cup \left(\frac{1}{2}, \infty\right) \) and \( (-\infty, -\frac{1}{2}) \cup \left(\frac{1}{2}, \infty\right) \) is open in \( \mathbb{R} \).
\( C = (-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1) \) is not open in \( \mathbb{R} \). It is also not open in \( Y \) because \( Y \) is a
metric space and no ball in \( Y \) with center \( \frac{1}{2} \) and positive radius is contained in \( C \).
\( D = [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1] \) is not open in \( \mathbb{R} \). It is also not open in \( Y \) for the same reason as above.

\( E = (-1, 0) \cup [(0, 1) - \{ \frac{1}{n} : n \in \mathbb{Z}_+ \}] \). \( (0, 1) - \{ \frac{1}{n} : n \in \mathbb{Z}_+ \} \) is open in \( \mathbb{R} \)
because if \( x \in (0, 1) - \{ \frac{1}{n} : n \in \mathbb{Z}_+ \} \), then \( \frac{1}{m+1} < x < \frac{1}{m} \) for some \( m \in \mathbb{Z}_+ \).
We can then choose an \( \epsilon' > 0 \) such that \( \frac{1}{m+1} < x - \epsilon < x + \epsilon < \frac{1}{m} \), and hence \( (x-\epsilon, x+\epsilon) \subset (0,1) - \{ \frac{1}{n} : n \in \mathbb{Z}_+ \} \). Since \( E \) is open in \( \mathbb{R} \), it’s open in \( Y \).

§16.9. Let \( T \) denote the dictionary order topology on \( \mathbb{R} \times \mathbb{R} \); it has basis
\[ \mathcal{B} = \{ (a \times b, c \times d) : a < c, \text{ or } a = c \text{ and } b < d \} \]
(see §14, Example 2). Let \( T' \) denote the topology on \( \mathbb{R}_d \times \mathbb{R} \); it has basis
\[ \mathcal{B}' = \{ \{x\} \times (c, d) : c < d \} \].
Suppose \( u \times v \in (a \times b, c \times d) \), where \( (a \times b, c \times d) \in \mathcal{B} \). We show that there is a
\( B' \in \mathcal{B}' \) with \( u \times v \in B' \subset (a \times b, c \times d) \).

There are several cases to consider.

Case 1. \( a = c \). Then \( (a \times b, c \times d) = a \times (b, d) \) which is already in \( \mathcal{B}' \).

Case 2. \( u = a \) and \( a < c \). Then
\[ u \times v \in \{a\} \times (b, v+1) \subset (a \times b, c \times d) \].

Case 3. \( a < u < c \). Then
\[ u \times v \in \{u\} \times (v-1, v+1) \subset (a \times b, c \times d) \].

Case 4. \( u = c \) and \( a < c \). Then
\[ u \times v \in \{c\} \times (v-1, d) \subset (a \times b, c \times d) \].

In any event, such a \( B' \) can be found; this proves that \( T \subset T' \) (Lemma 13.3).
On the other hand, any element of \( \mathcal{B}' \) is an element of \( \mathcal{B} \); hence \( T' \subset T \). Putting
these together gives \( T = T' \). Finally, note that \( T' \) is finer than the standard topology on \( \mathbb{R} \times \mathbb{R} \), since the
standard topology has \( \{ (a, b) \times (c, d) : a < b, c < d \} \) as a basis, and
\[ (a, b) \times (c, d) = \bigcup_{x \in (a, b)} \{x\} \times (c, d) \]
is open in \( T' \). But \( T' \) is not the same as the standard topology; for example, \( \{x\} \times (a, b) \) with \( a < b \) is in \( T' \) but is not open in the standard topology.
§17.6. a. If $A \subseteq B$, then $A \subseteq \overline{B}$. Since $\overline{A}$ is the intersection of all closed subsets containing $A$, and $\overline{B}$ is closed, it therefore follows that $\overline{A} \subseteq \overline{B}$.

b. Since $A \subseteq A \cup B$, we have $\overline{A} \subseteq \overline{A \cup B}$. Similarly, $\overline{B} \subseteq \overline{A \cup B}$, so $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. Now $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$, and $\overline{A \cup B}$ is the intersection of all closed subsets containing $A \cup B$; hence $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. Therefore, $\overline{A} \cup \overline{B} = \overline{A \cup B}$.

c. Since $A_\beta \subseteq \bigcup A_\alpha$, we have $\overline{A_\beta} \subseteq \bigcup \overline{A_\alpha}$, and therefore, $\bigcup \overline{A_\alpha} \subseteq (\bigcup \overline{A_\alpha})$. Equality does not hold in general, however. For example, let $X = \mathbb{R}$ with the standard topology, and let

$$A_n = (-\infty, -\frac{1}{n}] \cup [\frac{1}{n}, \infty)$$

for $n \in \mathbb{Z}_+$. Then $A_n$ is closed, so

$$\bigcup \overline{A_n} = \bigcup A_n = \mathbb{R} - \{0\},$$

but

$$\overline{\bigcup A_n} = \mathbb{R} - \{0\} = \mathbb{R}.$$

§17.13. First suppose that $X$ is Hausdorff. We will show that if $(u, v) \notin \Delta$, then $(u, v) \notin \overline{\Delta}$. Indeed, $u \neq v$, so there exist neighborhoods $U$ of $u$ and $V$ of $v$ in $X$ such that $U \cap V = \emptyset$. This means that $(U \times V) \cap \Delta = \emptyset$. Since $U \times V$ is open in $X \times X$, it follows that $(u, v) \notin \overline{\Delta}$.

Conversely, suppose that $\Delta$ is closed in $X \times X$. Then, if $u \neq v$, there exists a neighborhood $W$ of $(u, v)$ in $X \times X$ such that $W \cap \Delta = \emptyset$. But $\{U \times V : U, V \text{ open in } X\}$ is a basis for the topology on $X \times X$; hence there exist open sets $U$ and $V$ such that $(u, v) \in U \times V \subseteq W$. This means that $u \in U$, $v \in V$, and, since $U \times V \cap \Delta = \emptyset$, that $U \cap V = \emptyset$. Therefore, $X$ is Hausdorff.

Extra Exercise: a. Given any $\epsilon > 0$, there exists $n \in \mathbb{Z}_+$ with $\frac{1}{n} < \epsilon$. This implies that $(-\epsilon, \epsilon) \cap K \neq \emptyset$. Since any neighborhood of 0 contains $(-\epsilon, \epsilon)$ for some $\epsilon > 0$, it therefore follows that $0 \in K$. On the other hand, if $x < 0$, $(-\infty, 0)$ is a neighborhood of $x$ whose intersection with $K$ is empty. Thus $x \notin K$. If $x > 0$, but $x \notin K$, then there exists an integer $m \geq 0$ such that $\frac{1}{m+1} < x < \frac{1}{m}$, where we set $\frac{1}{0} = \infty$. But $(\frac{1}{m+1}, \frac{1}{m}) \cap K = \emptyset$, so $x \notin K$. From all this we conclude that $K = K \cup \{0\}$.

b. Let $x \in \mathbb{R}$, and let $U$ be any neighborhood of $x$. Then $\mathbb{R} - U$ is finite, so $U$ contains all but a finite number of elements of $\mathbb{R}$. But $K$ is infinite; therefore $U \cap K \neq \emptyset$. This implies that $x \in K$, and, since $x$ was arbitrary, that $K = \mathbb{R}$.

c. By an argument similar to what we did in class, the upper limit topology is finer than the standard topology. Hence the closure $\overline{K}$ of $K$ in the upper limit topology is a subset of the closure of $K$ in the standard topology. By part a), this implies that $\overline{K} \subseteq K \cup \{0\}$. But $0 \notin \overline{K}$, since $(-1, 0]$ is a neighborhood of 0 with $(-1, 0] \cap K = \emptyset$. Therefore, $\overline{K} = K$. 

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