§28.2. The set $A = \{ \frac{1}{2} - \frac{1}{n} : n \geq 2 \}$ does not have a limit point in $\mathbb{R}$. Indeed, if $x \geq \frac{1}{2}$, then $[x, \infty) \cap A = \emptyset$, and if $x < 0$, then $(-\infty, 0] \cap A = \emptyset$. If $0 \leq x < \frac{1}{2}$, then $\frac{1}{2} - \frac{1}{n} \leq x < \frac{1}{2} - \frac{1}{n+1}$ for some $n$. Then $[x, \frac{1}{2} - \frac{1}{n+1}] \cap A \subset \{x\}$.

§29.1. Suppose $C \subset \mathbb{Q}$ contains a nonempty open set in $\mathbb{Q}$. Then $(a, b) \cap \mathbb{Q} \subset C$ for some $a, b \in \mathbb{R}$ with $a < b$. Now regard $C$ as a subspace of $\mathbb{R}$. It inherits the same topology as it does as a subspace of $\mathbb{Q}$. But any irrational number $r$ with $a < r < b$ is a limit point of $(a, b) \cap \mathbb{Q}$. Since $r \notin C$, it follows that $C$ is not closed in $\mathbb{R}$ and thus cannot be compact. Therefore, $\mathbb{Q}$ is not locally compact.

§29.11 a. Let $A \subset Y \times Z$ and suppose that $\pi^{-1}(A)$ is open. We must show that, if $\pi \times y \in A$, then there exists a neighborhood of $\pi \times y$ (in $Y \times Z$) contained in $A$. To this end, suppose that $\pi(x \times y) = \pi \times y$. We continue by following the hint. Since $\pi^{-1}(A)$ is assumed to be open, there exist open sets $U_1$ and $W$ such that $x \times y \in U_1 \times W \subset \pi^{-1}(A)$.

Now $Z$ is locally compact Hausdorff; thus there exists a neighborhood $V$ of $y$ with $\overline{V}$ compact and $V \subset W$. We then have $x \times y \in U_1 \times V \subset U_1 \times \overline{V} \subset \pi^{-1}(A)$.

Next observe that, if $U_i \times \overline{V} \subset \pi^{-1}(A)$, then $p(U_i) \times \overline{V} \subset \pi^{-1}(A)$, and $U_i \times \overline{V} \subset p^{-1}(p(U_i)) \times \overline{V} = \pi^{-1}(p(U_i) \times \overline{V}) \subset \pi^{-1}(A)$.

For each point $w \in p^{-1}(p(U_i))$, we then have $\{w\} \times \overline{V} \subset \pi^{-1}(A)$; hence by the tube lemma, there exists a neighborhood $U_w$ of $w$ such that $U_w \times \overline{V} \subset \pi^{-1}(A)$. Set $U_{i+1} = \bigcup_{w \in p^{-1}(p(U_i))} U_w$.

$U_{i+1} \supset p^{-1}(p(U_i))$ and $U_{i+1} \times \overline{V} \subset \pi^{-1}(A)$. Let $U = \bigcup_i U_i$. Then $U \subset p^{-1}(p(U)) = \bigcup_i p^{-1}(p(U_i)) \subset \bigcup_i U_{i+1} = U$;

this implies that $p^{-1}(p(U)) = U$. Since $p$ is a quotient map, and the inverse image of $p(U)$ is open, $p(U)$ must be open. By construction, $U \times V \subset \pi^{-1}(A)$; thus $p(U) \times V \subset A$. $p(U) \times V$ is therefore a neighborhood of $\pi \times y$ contained in $A$, proving that $A$ is open and that $\pi$ is in fact a quotient map.

b. Observe that $p \times q = (i_B \times q) \circ (p \times i_C)$. By part a), $i_B \times q$ and $p \times i_C$ are quotient maps. Since the composition of two quotient maps is a quotient map, so is $p \times q$.
Extra Exercise. First assume that $f$ is proper. If $V$ is open in $Y^*$, then $V$ is either an open set in $Y$ (if $\infty \notin V$) or $V = Y^* \setminus C$ where $C$ is a compact subset of $Y$ (if $\infty \in V$). If $V$ is an open set in $Y$, then $(f^*)^{-1}(V) = f^{-1}(V)$, which is an open subset of $X$ since $f$ is continuous. If $V = Y^* \setminus C$, then
\[(f^*)^{-1}(V) = (f^*)^{-1}(Y^* \setminus C) = X^* \setminus f^{-1}(C).\]
But $f$ is proper, so $f^{-1}(C)$ is a compact subspace of $X$. Hence by the definition of the topology on $X^*$, $(f^*)^{-1}(V)$ is open in $X^*$, and therefore $f^*$ is continuous.

Conversely, suppose that $f^*$ is continuous. If $C$ is a compact subspace of $Y$, then $Y^* \setminus C$ is an open subset of $Y^*$ and hence $(f^*)^{-1}(Y^* \setminus C)$ is open in $X^*$. But $(f^*)^{-1}(Y^* \setminus C) = X^* \setminus f^{-1}(C)$; by the definition of the topology on $X^*$, we must have that $f^{-1}(C)$ is compact. This proves that $f$ is proper.