Point totals are shown in parentheses.

(10) 1. Let \((X, d)\) be a metric space, let \(a \in X\), and let \(r > 0\). Define
\[
\overline{B}(a, r) \equiv \{x \in X : d(a, x) \leq r\}.
\]

a. Prove that \(\overline{B}(a, r)\) is a closed subset of \(X\) (with the topology induced by the metric \(d\)).

b. Give an example of a metric space where \(\overline{B}(a, r)\) is not always the closure of \(B(a, r)\). (Recall that \(B(a, r) = \{x \in X : d(a, x) < r\}\).)

a. \(X - \overline{B}(a, r) = \{x \in X : d(x, a) > r\}\). We will show that \(X - \overline{B}(a, r)\) is open. If \(y \in X - \overline{B}(a, r)\), let \(\varepsilon = d(a, y) - r > 0\). Since
\[
d(a, y) \leq d(a, x) + d(x, y),
\]
we have that
\[
d(a, x) \geq d(a, y) - d(x, y).
\]
Thus \(d(x, y) < \varepsilon \Rightarrow d(a, x) > r\), so \(B(y, \varepsilon) \subset X - \overline{B}(a, r)\). This implies that \(X - \overline{B}(a, r)\) is open.

b. Let \(X\) be a set with at least 2 elements and let \(d\) be the discrete metric on \(X\). Then \(\overline{B}(a, 1) = X\) for any \(a \in X\). But \(B(a, 1) = \{a\}\) and \(\{a\}\) is closed in \(X\), so the closure of \(B(a, 1)\) is just \(\{a\}\).
3. Let $\mathbb{R}_e$ denote $\mathbb{R}$ with the lower limit topology; recall that a basis for this topology is $\{(a, b) : a < b\}$.

a. Prove that there exists a countable subset $A$ whose closure in $\mathbb{R}_e$ is $\mathbb{R}$.

b. Prove that there does not exist a basis for $\mathbb{R}_e$ containing only a countable number of sets.

Extra Credit: Prove that parts a) and b) imply that $\mathbb{R}_e$ is not metrizable.

a. Let $A = \mathbb{Q}$. To prove that the closure of $A$ in $\mathbb{R}_e$ is $\mathbb{R}$, it suffices to show that, for each $x \in \mathbb{R}$ and each basic open set $[a, b)$ with $x \in (a, b)$, we have $[a, b) \cap \mathbb{Q} \neq \emptyset$. But this is clear, since there is a rational number between any two real numbers.

b. Suppose $B$ is a basis for $\mathbb{R}_e$. Then for each $a, b \in \mathbb{R}$,

$$[a, b) = \bigcup_{\alpha} B_{\alpha}$$

for some collection of sets $\{B_{\alpha}\}$ in $B$. Notice that $a$ must be in $B_{\alpha}$ for some $\alpha$ and must in fact be the smallest element of $B_{\alpha}$. Thus, for each $a \in \mathbb{R}$, there exists $B_a \in B$ such that $a$ is the smallest element of $B_a$. We then get an injective function from $\mathbb{R}$ to $B$ sending $a$ to $B_a$. But $\mathbb{R}$ is uncountable; therefore $B$ must also be uncountable.

Extra Credit (brief sketch): If $X$ is a metric space with a countable subset $A = \{x_i : i \in \mathbb{N}\}$ whose closure is $X$, then you can check that $B = \{B(x_i, r) : x_i \in A, r \in \mathbb{Q}_+\}$ is a countable basis for the topology on $X$ induced by this metric. Since $\mathbb{R}_e$ has a countable subset $A$ but no countable basis, it cannot be metrizable.