1-5.1. a. $d\alpha/ds = \left(-(a/c) \sin(s/c) ,(a/c) \cos(s/c), b/c\right)$ and thus

$$|\alpha'(s)| = \sqrt{\frac{a^2}{c^2} \sin^2 \frac{s}{c} + \frac{a^2}{c^2} \cos^2 \frac{s}{c} + \frac{b^2}{c^2}}$$

$$= \sqrt{\frac{a^2 + b^2}{c^2}} = 1,$$

since $c^2 = a^2 + b^2$. Therefore $\alpha$ is parametrized by arc length.

b. $\alpha''(s) = \left(-\frac{(a/c)^2}{c} \cos\left(\frac{s}{c}\right), -\frac{(a/c)^2}{c} \sin\left(\frac{s}{c}\right), 0\right)$ so $k(s) = \frac{a}{c^2}$. To compute the torsion, start with

$$b(s) = t(s) \land n(s) = \left(-\frac{a}{c} \sin \frac{s}{c}, \frac{a}{c} \cos \frac{s}{c} - \frac{b}{c} \cos \frac{s}{c}, \frac{b}{c} \sin \frac{s}{c}, 0\right)$$

$$= \left(-\frac{a}{c} \sin \frac{s}{c}, \frac{a}{c} \cos \frac{s}{c} - \frac{b}{c} \cos \frac{s}{c}, \frac{b}{c} \sin \frac{s}{c}, 0\right) \land \left(-\cos \frac{s}{c}, -\sin \frac{s}{c}, 0\right)$$

$$= \left(-\frac{b}{c} \sin \frac{s}{c}, -\frac{b}{c} \cos \frac{s}{c}, \frac{a}{c}\right).$$

Then $b'(s) = \left(\frac{(b/c)^2}{c^2} \cos(s/c), \frac{b/c^2}{c^2} \sin(s/c), 0\right) = -\left(\frac{b}{c^2}\right)n(s)$, so that $\tau(s) = \frac{b}{c^2}$.

c. The plane passing through $\alpha(s) = (a \cos(s/c), a \sin(s/c), bs/c)$ with normal $b(s) = ((b/c) \sin(s/c), -(b/c) \cos(s/c), a/c)$ is the osculating plane; it is given by the set of $(x, y, z)$ with

$$\frac{b}{c} \sin \frac{s}{c} \cdot \left(x - a \cos \frac{s}{c}\right) - \frac{b}{c} \cos \frac{s}{c} \cdot \left(y - a \sin \frac{s}{c}\right) + \frac{a}{c} \left(z - \frac{bs}{c}\right) = 0.$$

Simplify this expression to get

$$\left(\frac{b}{c} \sin \frac{s}{c}\right) x - \left(\frac{b}{c} \cos \frac{s}{c}\right) y + \frac{a}{c} z = \frac{bas}{c^2}.$$

d. Since $n(s) = \left(-\cos(s/c), -\sin(s/c), 0\right)$, it follows that the line through $\alpha(s)$ in the direction of $n(s)$ is parallel to the $xy$-plane. It also intersects the $z$-axis at the point $(0, 0, bs/c)$ and must therefore do so at an angle of $\pi/2$.

e. $e_3 \cdot (\alpha'(s)/|\alpha'(s)|) = \cos \theta$, where $\theta$ is the angle between $\alpha'(s)$ and the $z$-axis. Since

$$e_3 \cdot \frac{\alpha'(s)}{|\alpha'(s)|} = \frac{b}{c},$$

it follows that this angle is constant.
1-5.5. a. Without loss of generality, we may assume that \( \alpha \) is parametrized by arc length; we may also assume, by translating if necessary, that the fixed point through which all the tangent lines pass is the origin. This implies that the line through the origin and \( \alpha(s) \) is the tangent line of \( \alpha \) at \( s \); hence

\[
\alpha(s) = c(s)\alpha'(s)
\]

for some scalar \( c(s) \). In fact, \( c(s) = \alpha(s) \cdot \alpha'(s) \), so that \( c \) is a differentiable function of \( s \). Now differentiate both sides of (1) and rearrange to get

\[
c(s)\alpha''(s) = (1 - c'(s))\alpha'(s).
\]

But \( \alpha''(s) \) is perpendicular to \( \alpha'(s) \) (since \( \alpha \) is parametrized by arc length); the above formula then implies that \( c(s)\alpha''(s) = 0 \) for all \( s \). I claim that in fact \( \alpha''(s) = 0 \) for all \( s \) and hence, by a previous homework exercise (1-2.3), the trace of \( \alpha \) is an infinitely differentiable function on \( \mathbb{R} \). (The factor \( e^{-\frac{1}{t^2}} \) is introduced so that \( \alpha \) is (infinitely) differentiable. Recall that the function

\[
f(t) = \begin{cases} 
0 & t = 0 \\
e^{-\frac{1}{t^2}} & t \neq 0 
\end{cases}
\]

is an infinitely differentiable function on \( \mathbb{R} \) all of whose derivatives vanish at 0.) However, the trace of \( \alpha \) is the graph of the function \( y = |x| \) for \( |x| < 1 \), yet all of the tangent lines of \( \alpha \) pass through the origin.

1-5.6. a. Let \( \rho : \mathbb{R}^3 \to \mathbb{R}^3 \) be an orthogonal transformation, and let \( u, v \) be elements of \( \mathbb{R}^3 \). We have

\[
|u|^2 = u \cdot u = \rho(u) \cdot \rho(u) = |\rho(u)|^2,
\]

so the norm of a vector is invariant under an orthogonal transformation. Similarly, let \( \theta_0 \) be the angle between \( u \) and \( v \), and let \( \theta_1 \) be the angle between \( \rho(u) \) and \( \rho(v) \), \( 0 \leq \theta_i \leq \pi \). Then

\[
|u||v| \cos \theta_0 = u \cdot v = \rho(u) \cdot \rho(v) = |\rho(u)||\rho(v)| \cos \theta_1.
\]

Since \( |u| = |\rho(u)| \) and \( |v| = |\rho(v)| \), we have that \( \cos \theta_0 = \cos \theta_1 \) (since we must have \( u \) and \( v \) nonzero to talk about the angle between them) and hence \( \theta_0 = \theta_1 \).
b. Let \( \rho : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be an orthogonal transformation with positive determinant and let \( u,v \in \mathbb{R}^3 \). If \( u \) and \( v \) are collinear, then \( u \wedge v = 0 \) and hence \( \rho(u \wedge v) = 0 \). On the other hand, \( \rho(u) \) and \( \rho(v) \) are also collinear, so \( \rho(u) \wedge \rho(v) = 0 = \rho(u \wedge v) \).

Now suppose that \( u \) and \( v \) are not collinear. Then \( \rho(u) \) and \( \rho(v) \) are not collinear (why?), and \( \rho(u) \wedge \rho(v) \) is the unique vector of length \( |\rho(u)||\rho(v)| \sin \theta_1 \) perpendicular to both \( \rho(u) \) and \( \rho(v) \) and such that \( (\rho(u), \rho(v), \rho(u) \wedge \rho(v)) \) is positively oriented. Once again, \( \theta \) is the angle between \( \rho(u) \) and \( \rho(v) \). We will show that \( \rho(u \wedge v) \) has these same properties. Indeed, \( u \wedge v \) is perpendicular to both \( u \) and \( v \); hence \( \rho(u \wedge v) \) is perpendicular to \( \rho(u) \) and \( \rho(v) \). Since \( (u,v,u \wedge v) \) is positively oriented and \( \rho \) has positive determinant, so is \( (\rho(u), \rho(v), \rho(u \wedge v)) \). Moreover, by part a,

\[
|\rho(u \wedge v)| = |u \wedge v| = |u||v| \sin \theta_0 = |\rho(u)||\rho(v)| \sin \theta_1;
\]

therefore \( \rho(u \wedge v) = \rho(u) \wedge \rho(v) \).

This proof also shows that if \( \rho \) is orthogonal with negative determinant, then \( \rho(u) \wedge \rho(v) = -\rho(u \wedge v) \).

c. It is trivial to check that the arc length, curvature, and torsion are invariant under a translation.

Now let \( \rho : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be an orthogonal transformation with positive determinant, and let \( \alpha : I \rightarrow \mathbb{R}^3 \) be a parametrized curve. Consider also the curve \( \rho \circ \alpha : I \rightarrow \mathbb{R}^3 \). Since \( \rho \) is a linear transformation, it follows that

\[(\rho \circ \alpha)'(t) = \rho(\alpha'(t)),\]

so

\[(\rho \circ \alpha')(t) = |\rho(\alpha'(t))| = |\alpha'(t)|\]

by the orthogonality of \( \rho \). It is clear from this that the arc length of \( \rho \circ \alpha \) from \( t_0 \) to \( t \) is the same as the arc length of \( \alpha \) from \( t_0 \) to \( t \). Moreover, if \( \alpha \) is parametrized by arc length, so is \( \rho \circ \alpha \). In such a case, \( k(s) \), the curvature of \( \alpha \), is given by

\[k(s) = |\alpha''(s)|,\]

and \( k_\rho(s) \), the curvature of \( \rho \circ \alpha \), is given by

\[k_\rho(s) = |(\rho \circ \alpha)''(s)|.\]

But \( (\rho \circ \alpha)' = \rho \circ \alpha' \), so the same arguments as in (1) and (2) show that

\[(\rho \circ \alpha)''(s) = \rho(\alpha''(s))\]

and

\[|(\rho \circ \alpha)''(s)| = |\rho(\alpha''(s))| = |\alpha''(s)|.\]

Thus \( k(s) = k_\rho(s) \).

Finally, let \( t(s) \), \( n(s) \), \( b(s) \), and \( \tau(s) \) be defined for \( \alpha \) as in do Carmo; and let \( t_\rho(s) \), \( n_\rho(s) \), \( b_\rho(s) \), and \( \tau_\rho(s) \) be the corresponding functions for \( \rho \circ \alpha \). We have already shown that

\[t_\rho(s) = \rho(t(s))\]
\[n_\rho(s) = \rho(n(s));\]

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hence, by part a),
\[ b_\rho(s) = t_\rho(s) \wedge n_\rho(s) = \rho(t(s)) \wedge \rho(n(s)) = \rho(t(s) \wedge n(s)) = \rho(b(s)). \]

It then follows as in (1) that
\[ b'_\rho(s) = \rho(b'(s)), \]
so that
\[ b'_\rho(s) = \rho(\tau(s)n(s)) = \tau(s)\rho(n(s)) = \tau(s)n_\rho(s) \]
and therefore \( \tau_\rho(s) = \tau(s). \)

1-5.9. Let \( \alpha(s) \) be as stated in the problem. Then \( \alpha'(s) = (\cos \theta(s), \sin \theta(s)) \), so \( \alpha \) is parametrized by arc length. Moreover,
\[ \alpha''(s) = \theta'(s)(-\sin \theta(s), \cos \theta(s)) = k(s)(-\sin \theta(s), \cos \theta(s)), \]
and \((\cos \theta(s), \sin \theta(s)), (-\sin \theta(s), \cos \theta(s))\) is positively oriented. Thus the curvature of \( \alpha \) at \( s \) is \( k(s) \).

Now suppose \( \beta : I \rightarrow \mathbb{R}^2 \) is a curve parametrized by arc length whose curvature is the given function \( k(s) \). Since \( |\beta'(s)| = 1 \), we may follow the hint to write
\[ \beta'(s) = (\cos \psi(s), \sin \psi(s)) \]
for some differentiable function \( \psi : I \rightarrow \mathbb{R} \). Then
\[ \beta''(s) = \psi'(s)(-\sin \psi(s), \cos \psi(s)), \]
and since \((\cos \psi(s), \sin \psi(s)), (-\sin \psi(s), \cos \psi(s))\) is positively oriented, it follows that \( \psi'(s) = k(s) \). Thus
\[ \psi(s) = \int k(s) \, ds + \omega \]
for some constant \( \omega \). Moreover, \( \beta \) is an antiderivative of \( \beta' \), so
\[ \beta(s) = \left( \int \cos \psi(s) \, ds + u, \int \sin \psi(s) \, ds + v \right) \]
for some constants \( u \) and \( v \).

If \( u \) and \( v \) are changed, the curve \( \beta \) is translated, and if \( \omega \) is changed, \( \beta' \) is rotated. This implies that its antiderivative is rotated (and perhaps translated).

1-5.12. Define \( \phi(t) = \int_{t_0}^t |\alpha'(u)| \, du \) and \( \beta = \alpha \circ \phi^{-1} \).

a. This problem asks you to compute \( d(\phi^{-1})/ds \) and \( d^2(\phi^{-1})/ds^2 \). First of all,
\[ \frac{d(\phi^{-1})}{ds} = \frac{1}{\phi'(\phi^{-1}(s))} = \frac{1}{|\alpha'(\phi^{-1}(s))|}. \]

Now, since \( |\alpha'(t)| = (\alpha'(t) \cdot \alpha'(t))^\frac{1}{2} \), we have
\[ \frac{d}{dt}(|\alpha'(t)|) = \frac{\alpha'(t) \cdot \alpha''(t)}{|\alpha'(t)|^2}, \]
and therefore
so
\[
\frac{d}{dt} \left( \frac{1}{|\alpha'(t)|} \right) = -\frac{\alpha'(t) \cdot \alpha''(t)}{|\alpha'(t)|^3}.
\]
Therefore,
\[
\frac{d^2 \phi^{-1}}{ds^2} = -\frac{\alpha'(\phi^{-1}(s)) \cdot \alpha''(\phi^{-1}(s))}{|\alpha'(\phi^{-1}(s))|^3} \quad \text{and} \quad \frac{d \phi^{-1}}{ds} = -\frac{\alpha'(\phi^{-1}(s)) \cdot \alpha''(\phi^{-1}(s))}{|\alpha'(\phi^{-1}(s))|^4}.
\]

b. We need to compute the curvature of \( \beta \) at \( s \), where \( t = \phi^{-1}(s) \). Now
\[
\beta'(s) = \alpha'(\phi^{-1}(s)) \cdot \frac{d \phi^{-1}}{ds} = \frac{\alpha'(t)}{|\alpha'(t)|}.
\]
Hence
\[
\beta''(s) = \frac{d}{dt} \left( \frac{\alpha'(t)}{|\alpha'(t)|} \right) = \frac{d}{ds} \left( \frac{\alpha'(t)}{|\alpha'(t)|} \right) = \left( \frac{\alpha''(t)}{|\alpha'(t)|} - \frac{(\alpha'(t) \cdot \alpha''(t))\alpha'(t)}{|\alpha'(t)|^3} \right) \cdot \frac{1}{|\alpha'(t)|}
\]
\[= \frac{(\alpha'(t) \cdot \alpha'(t))\alpha''(t) - (\alpha'(t) \cdot \alpha''(t))\alpha'(t)}{|\alpha'(t)|^4}.\]

In this expression, the dot product of the numerator with itself is given by
\[
|\alpha''(t)|^2 |\alpha'(t)|^4 - 2(\alpha'(t) \cdot \alpha''(t))^2 |\alpha'(t)|^2 + |\alpha'(t)|^2 (\alpha'(t) \cdot \alpha''(t))^2,
\]
which simplifies to
\[
|\alpha'(t)|^2 (|\alpha''(t)|^2 |\alpha'(t)|^2 - (\alpha'(t) \cdot \alpha''(t))^2).
\]
But
\[
|\alpha''(t)|^2 |\alpha'(t)|^2 - (\alpha'(t) \cdot \alpha''(t))^2 = |\alpha'(t) \wedge \alpha''(t)|^2,
\]
where we have used the formula
\[
(u \wedge v) \cdot (u \wedge v) = \left| \begin{array}{cccc} u \cdot u & u \cdot v \\ u \cdot v & v \cdot v \end{array} \right|.
\]
Thus
\[
|\beta''(s)|^2 = \frac{|\alpha'(t)|^2 |\alpha'(t) \wedge \alpha''(t)|^2}{|\alpha'(t)|^8},
\]
so
\[
|\beta''(s)| = \frac{|\alpha'(t) \wedge \alpha''(t)|}{|\alpha'(t)|^3}.
\]
This is the curvature of \( \beta \) at \( s \).

c. Let \( b_\beta(s) \) be the binormal vector of \( \beta \) at \( s \), where \( t = \phi^{-1}(s) \). Then
\[
b_\beta(s) = \beta'(s) \wedge \frac{\beta''(s)}{|\beta''(s)|} = \frac{\alpha'(t) \wedge \alpha''(t)}{|\alpha'(t) \wedge \alpha''(t)|}.\]
Now write
\[ \frac{d}{dt} \left( \frac{1}{\alpha'(t) \wedge \alpha''(t)} \right) = \frac{g(t)}{|\alpha'(t) \wedge \alpha''(t)|^3} \]
with
\[ g(t) = - (\alpha'(t) \wedge \alpha''(t)) \cdot (\alpha'(t) \wedge \alpha'''(t)). \]
Then
\[ b'_{\beta}(s) = \left[ \frac{\alpha'(t) \wedge \alpha''(t)}{|\alpha'(t) \wedge \alpha''(t)|} + \frac{g(t)(\alpha'(t) \wedge \alpha''(t))}{|\alpha'(t) \wedge \alpha''(t)|^3} \right] \frac{dt}{ds} \]
\[ = \left[ \frac{\alpha'(t) \wedge \alpha''(t)}{|\alpha'(t) \wedge \alpha''(t)|} + \frac{g(t)(\alpha'(t) \wedge \alpha''(t))}{|\alpha'(t) \wedge \alpha''(t)|^3} \right] \frac{1}{|\alpha'(t)|}. \]

Let \( \tau_{\beta}(s) \) be the torsion of \( \beta \) at \( s \). Then
\[ \tau_{\beta}(s) = \frac{b'_{\beta}(s) \cdot \beta''(s)}{|\beta''(s)|} \]
\[ = \frac{1}{|\beta''(s)||\alpha'(t)|} \left( (\alpha'(t) \wedge \alpha''(t)) \cdot \alpha''(t) \right), \]

since \( \alpha'(t) \wedge \alpha''(t) \) is orthogonal to \( \alpha'(t) \), and \( \alpha'(t) \wedge \alpha''(t) \) is orthogonal to both \( \alpha'(t) \) and \( \alpha''(t) \). Now
\[ (\alpha'(t) \wedge \alpha''(t)) \cdot \alpha''(t) = - (\alpha'(t) \wedge \alpha''(t)) \cdot \alpha'''(t) \]
(why?); we therefore obtain
\[ \tau_{\beta}(s) = - \frac{(\alpha'(t) \wedge \alpha''(t)) \cdot \alpha'''(t)}{|\alpha'(t) \wedge \alpha''(t)|^2}. \]

d. In terms of \( x(t) \) and \( y(t) \), we have that
\[ \beta'(s) = \frac{(x'(t), y'(t))}{(x'(t)^2 + y'(t)^2)^{\frac{1}{2}}}, \]
so that the normal vector is given by
\[ n(s) = (x'(t)^2 + y'(t)^2)^{-\frac{1}{2}}(-y'(t), x'(t)). \]

But, from part b),
\[ \beta''(s) = \frac{x''(x')^2 + x''(y')^2 - (x')^2 x'' - x'y'y'' + y''(x')^2 + y''(y')^2 - y'x'x'' - (y')^2 y''}{[(x')^2 + (y')^2]^2} \]
\[ = \frac{(x''y')^2 - x'y'y'' + y''(x')^2 - y'x'x''}{[(x')^2 + (y')^2]^2} \]
\[ = [(x')^2 + (y')^2]^{-\frac{1}{2}} (x'y'' - x''y') n(s). \]

This proves that the signed curvature of \( \beta \) at \( s \) is
\[ (x'(t)^2 + y'(t)^2)^{-\frac{1}{2}} (x'(t)y''(t) - x''(t)y'(t)). \]
We may assume that $\alpha$ is parametrized by arc length. At $t_0$, the function $f(t) = \alpha(t) \cdot \alpha(t)$ attains a maximum. Therefore, $f'(t_0) = 0$ and $f''(t_0) \leq 0$. Now

$$f'(t) = 2\alpha'(t) \cdot \alpha(t)$$

and

$$f''(t) = 2\alpha''(t) \cdot \alpha(t) + 2\alpha'(t) \cdot \alpha'(t)$$

$$= 2[(\alpha''(t) \cdot \alpha(t)) + 1],$$

since $\alpha$ is parametrized by arc length. We then have

$$\alpha''(t_0) \cdot \alpha(t_0) + 1 \leq 0,$$

so

$$|\alpha''(t_0) \cdot \alpha(t_0)| \geq 1.$$

But, by the Cauchy-Schwarz inequality,

$$|\alpha''(t_0) \cdot \alpha(t_0)| \leq |\alpha''(t_0)||\alpha(t_0)| = |k(t_0)||\alpha(t_0)|,$$

so

$$|k(t_0)| \geq \frac{1}{|\alpha(t_0)|}.$$