2-2.17. A set $C$ in $\mathbb{R}^n$ is a regular curve if for each $p \in C$, there exists an open set $I$ in $\mathbb{R}$, an open neighborhood $V$ of $p$ in $\mathbb{R}^n$, and a homeomorphism $x : I \to V \cap C$ such that

i. $x : I \to \mathbb{R}^n$ is smooth

ii. $x'(t) \neq \vec{0}$ for each $t \in I$. (This is the same as saying that $(d\mathbf{x})(t)$ is one-to-one.)

a. Let $C = f^{-1}\{a\}$, where $a$ is a regular value of $f$, and suppose $p = (x_0, y_0) \in C$. Assume that $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$. (The case $\frac{\partial f}{\partial x}(x_0, y_0)$ is similar.) Consider the function $F : U \to \mathbb{R}^2$ defined by

$$F(x, y) = (x, f(x, y)).$$

Then

$$\det(dF(p)) = \begin{vmatrix} \frac{\partial f}{\partial x}(p) & \frac{\partial f}{\partial y}(p) \\ 0 & 1 \end{vmatrix} = \frac{\partial f}{\partial y}(p) \neq 0,$$

so $dF(p)$ is invertible. By the inverse function theorem, there then exists a neighborhood $V$ of $p$ in $\mathbb{R}^2$ such that $W = F(V)$ is open and $F : V \to W$ is a diffeomorphism. Observe that $F^{-1} : W \cap \{u, v) : v = a\} \to V \cap C$ is a homeomorphism,
and let \( I = \{ u \mid (u, a) \in W \} \). Then \( j : I \rightarrow W \cap \{ (u, v) : v = a \} \) given by \( j(u) = (u, a) \) is a homeomorphism, so

\[ F^{-1} j : I \rightarrow W \cap C \]

is a homeomorphism as well. Since \( F^{-1} \) and \( j \) are both differentiable, so is \( F^{-1} j \). Finally,

\[ d(F^{-1} j)(t) = dF^{-1}(j(t)) \circ dj(t) \]

and both \( dF^{-1}(j(t)) \) and \( dj(t) \) are one-to-one. This proves that \( d(F^{-1} j)(t) \) is one-to-one for each \( t \in \mathbb{I} \) and therefore \( F^{-1} j \) is a parametrization. (Observe here that \( F^{-1}(u, v) = (u, g(u, v)) \), where \( g : W \rightarrow \mathbb{R} \) is smooth. Then

\[ (F^{-1} j)(t) = (t, g(t, a)), \]

so that \( W \cap C \) is the graph of the function whose value at \( t \) is \( g(t, a) \).)

... \( f^{-1} \{ a \} \) need not be connected. For example, let \( f(x) = x^2 \). Then \( 1 \) is a regular value, but \( f^{-1} \{ 1 \} \) consists of the lines \( x = 1 \) and \( x = -1 \) in \( \mathbb{R}^2 \).

b. Let \( a = (a_1, a_2) \) be a regular value of \( F_1 \) and let \( C = F^{-1} \{ a \} \subset \mathbb{R}^3 \). Suppose \( p = (x_0, y_0, z_0) \in C \).

Then

\[
\begin{bmatrix}
\frac{\partial F_1}{\partial x}(p) & \frac{\partial F_1}{\partial y}(p) & \frac{\partial F_1}{\partial z}(p) \\
\frac{\partial F_2}{\partial x}(p) & \frac{\partial F_2}{\partial y}(p) & \frac{\partial F_2}{\partial z}(p)
\end{bmatrix}
\]
is onto. This means that at least one of the Jacobian determinants

\[
\begin{align*}
\frac{\partial (F_1, F_2)}{\partial (x, y)}, & \quad \frac{\partial (F_1, F_2)}{\partial (x, z)}, & \quad \frac{\partial (F_1, F_2)}{\partial (y, z)}
\end{align*}
\]

must be nonzero at \( p \). Let us suppose that \( \frac{\partial (F_1, F_2)}{\partial (y, z)} (p) \neq 0 \); the other cases are similar.

Consider the function \( G : U \rightarrow \mathbb{R}^3 \) defined by

\[
G(x, y, z) = (x, F(x, y, z)).
\]

Then

\[
\det \left( dG(p) \right) = \begin{vmatrix}
1 & 0 & 0 \\
\frac{\partial F_1}{\partial x}(p) & \frac{\partial F_1}{\partial y}(p) & \frac{\partial F_1}{\partial z}(p) \\
\frac{\partial F_2}{\partial x}(p) & \frac{\partial F_2}{\partial y}(p) & \frac{\partial F_2}{\partial z}(p)
\end{vmatrix} = \frac{\partial (F_1, F_2)}{\partial (y, z)} (p) \neq 0;
\]

hence, by the inverse function theorem, there exists a neighborhood \( V \) of \( p \) in \( \mathbb{R}^3 \) such that \( W = G(V) \) is open in \( \mathbb{R}^3 \) and \( G : V \rightarrow W \) is a diffeomorphism. As before,

\[
G^{-1} : W \cap \{(u, v, w) : (v, w) = (a_1, a_2)\} \rightarrow V \cap C
\]

is a homeomorphism, and let \( I = \{u \mid (u, a_1, a_2) \in W\} \).
Then \( j: I \rightarrow W \cap \{(u,v,w): (v,w)=(a_1,a_2)\} \) given by \( j(u) = (u,a_1,a_2) \) is a homeomorphism, so

\[ G^{-1} \circ j: I \rightarrow V \cap C \]

is a homeomorphism as well. The same argument as in a) shows that \( G^{-1} \circ j \) is differentiable and that \( d(G^{-1} \circ j)(t) \) is one-to-one for each \( t \in I \); this then proves that \( G^{-1} \circ j \) is a parametrization.

Finally, note that \( C = S_1 \cap S_2 \), where \( S_1 = F_1^{-1}\{a_1\} \) and \( S_2 = F_2^{-1}\{a_2\} \). However, it need not be the case that \( a_1 \) is a regular value of \( F_1 \) and \( a_2 \) is a regular value of \( a_2 \), so we cannot conclude that \( S_1 \) and \( S_2 \) are always regular surfaces.

c. We first prove a preliminary result.

Lemma: Suppose \( C \) is a regular surface in \( \mathbb{R}^2 \) and \( p \in C \). Then there exists a neighborhood \( V \) of \( p \) in \( C \) such that \( V \) is the graph of a differentiable function of the form \( y = f(x) \) or \( x = g(y) \).

Proof: Let \( x: U_0 \rightarrow V_0 \) be a parametrization, with \( U_0 \) open in \( \mathbb{R} \), \( V_0 \) open in \( C \), \( p \in V_0 \), and \( x(q) = p \). Write \( x(t) = (x(t), y(t)) \). Since \( \frac{dx}{dt}(q) \neq 0 \), we
must have either \( x'(q) \neq 0 \) or \( y'(q) \neq 0 \). Suppose \( x'(q) \neq 0 \); the other case is similar. Then there exists a neighborhood \( U \) of \( q \) such that \( W = x(U) \) is open and \( x: U \to W \) is a diffeomorphism. Let \( V = x(U) \).

Since \( x \) is a homeomorphism and \( V_0 \) is open in \( C \), it follows that \( V \) is a neighborhood of \( p \) in \( C \). Moreover, \( x = \pi \circ x \), so \( \pi: V \to W \) is also a homeomorphism, where \( \pi \) is the projection onto the first coordinate. Therefore, given \( t \in W \), there exists a unique \( s \in \mathbb{R} \) such that \((t, s) \in V\). Define \( f: W \to \mathbb{R} \) by \( f(t) = s \). Then \( V \) is the graph of \( f \), and \( f \) is smooth since \( f = \pi_2 \circ x \circ x^{-1} \), where \( \pi_2 \) is the projection onto the second coordinate. This completes the proof.

Now let \( C = \{ (x, y) \in \mathbb{R}^2 : x^2 = y^3 \} \). (See Figure 1-2 on p. 3 of do Carmo for a picture.) I claim that there is no neighborhood \( V \) of \((0,0)\) in \( C \) which is the graph of a differentiable function \( y = f(x) \) or \( x = g(y) \). Certainly, we cannot have \( x = g(y) \), since both \((y^{3/2}, y)\) and \((-y^{3/2}, y)\) lie in \( V \) for \( y \) a sufficiently small positive number. If \( V \) is the graph of a function \( y = f(x) \), we must have \( f(x) = x^{2/3} \). But \( f \) is not differentiable at \( 0 \).
2-3.2. \( \pi \) is a map between surfaces. Using the definition of differentiability for such maps – and using the fact that \( \mathbb{R}^2 \) is parametrized by the map sending \((u,v)\) to \((u,v,0)\) – it suffices to show that \( \pi \circ x : U \to \mathbb{R}^2 \) is differentiable, where \( x : U \to S \) is a local parametrization of \( S \) and \( \pi \) is here the projection onto the first two coordinates. But
\[
(\pi \circ x)(u,v) = (x(u,v), y(u,v)),
\]
where \( x(u,v) = (x(u,v), y(u,v), z(u,v)) \). Since \( x(u,v) \) and \( y(u,v) \) are differentiable, so is \( \pi \circ x \).

14. Certainly, if \( A \) is open in \( S \), then \( A \) is a regular surface.

Conversely, suppose \( A \subset S \) and \( A \) is a regular surface. To show that \( A \) is open in \( S \), it suffices to show that whenever \( p \in A \), there exists a neighborhood of \( p \) in \( S \) completely contained in \( A \). To do this, start by taking \( x : U \to N \) to be a parametrization of \( A \) in a neighborhood of \( p \) and \( y : U' \to N' \) to be a parametrization of \( S \) in a neighborhood of \( p \). By shrinking \( U' \) if necessary, we may, as in the proof of Proposition 1 (pp.70-1), extend \( y \) to a diffeomorphism
$\gamma: V \to W$, where $V$ and $W$ are open in $\mathbb{R}^3$. (Here we regard $\mathbb{R}^2 \subset \mathbb{R}^3$ in the usual way.) Then, by shrinking $U$ if necessary, we get that

$\gamma^{-1} \circ \gamma = \gamma^{-1} \circ \gamma : U \to \mathbb{R}^2$

is differentiable. Moreover, for $q \in U$

$$d(\gamma^{-1} \circ \gamma)(q) = (d \gamma^{-1})(x(q)) \cdot dx(q)$$

is the composition of two one-to-one linear transformations and so is one-to-one. This implies that $d(\gamma^{-1} \circ \gamma)(q)$ is invertible for each $q \in U$, so by the inverse function theorem, $(\gamma^{-1} \circ \gamma)(U)$ is open in $\mathbb{R}^2$. Since $\gamma^{-1}$ is a homeomorphism from an open subset of $S$ to an open subset of $\mathbb{R}^2$, it follows that $N = x(U)$ is open in $S$. This completes the proof.