do Carmo

3. Let $\mathbf{x} : \mathbb{R}^2 \rightarrow \mathbb{S}^2 \cap \{N\}$ be given by

$$
\mathbf{x}(u,v) = \left( \frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{2(u^2+v^2)}{u^2+v^2+4} \right),
$$

where here $\mathbb{S}^2$ is the sphere of radius 1 with center at $(0,0,1)$ and $N=(0,0,2)$. (Of course, $\mathbf{x}$ is just $\pi^{-1}$ of exercise 16 in 2-2.) Now

$$
\begin{align*}
\mathbf{x}_u(u,v) &= \left( \frac{4(v^2-u^2+y)}{(u^2+v^2+y)^2}, -\frac{8uv}{(u^2+v^2+y)^2}, \frac{16u}{(u^2+v^2+y)^2} \right) \\
\mathbf{x}_v(u,v) &= \left( -\frac{8uv}{(u^2+v^2+y)^2}, \frac{4(u^2-v^2+y)}{(u^2+v^2+y)^2}, \frac{16v}{(u^2+v^2+y)^2} \right)
\end{align*}
$$

If $(u_0,v_0) \in \mathbb{R}^2$,

$$
\begin{align*}
\mathbf{F}(u_0,v_0) &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle_{\mathbb{S}^2}(u_0,v_0) = 0 \\
\mathbf{E}(u_0,v_0) &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle_{\mathbb{S}^2}(u_0,v_0) = \frac{16}{(u_0^2+v_0^2+y)^2} \\
\mathbf{G}(u_0,v_0) &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle_{\mathbb{S}^2}(u_0,v_0) = \frac{16}{(u_0^2+v_0^2+y)^2}
\end{align*}
$$

Therefore, if $p = \mathbf{x}(u_0,v_0)$, we have

$$
I_p(a \mathbf{x}_u + b \mathbf{x}_v) = \frac{16}{(u_0^2+v_0^2+y)^2} (a^2 + b^2).
$$
Il. a. Parametrize $C$ by arc length, as in Example 4 on p. 76 of de Carmo, we let

$$x = f(v), \quad z = g(v), \quad 0 < v < l, \quad f(v) > 0$$

be this parametrization. We then have the parametrization $\mathbf{x}: U \to S - C$ given by

$$\mathbf{x}(u,v) = (f(u)\cos u, \quad f(u)\sin u, \quad g(u)),$$

where $U = \{(u,v) \in \mathbb{R}^2 : 0 < u < 2\pi, \quad 0 < v < l\}$. Then the area of $S$ is given by

$$A(S) = \int_0^l \int_0^{2\pi} |\mathbf{x}_u \times \mathbf{x}_v| \, du \, dv.$$  

(Actually, this expression is not quite justified (why?), however, you need not prove a proof.)

Now

$$\mathbf{x}_u = (-f(u)\sin u, \quad f(u)\cos u, \quad 0)$$
$$\mathbf{x}_v = (f'(u)\cos u, \quad f'(u)\sin u, \quad g'(u)),$$

so

$$|\mathbf{x}_u \times \mathbf{x}_v| = \left| \begin{vmatrix} y' & z' & -f(u) \sin u \\ z' & x' & -f(u) \cos u \\ 0 & f(u) & f'(u) \end{vmatrix} \right|$$

$$= f(u) \sqrt{g''(u)^2 + f'(u)^2} = f(u) = \rho(u)$$

Hence,

$$A(S) = \int_0^l \int_0^{2\pi} \rho(u) \, du \, dv = 2\pi \int_0^l \rho(u) \, du.$$
b. Let \( C \) be the pictured circle, the surface obtained by rotating it around the z-axis is a torus \( T \). If we parametrize \( C \) by starting at \( x = a \pi r \) and proceeding counterclockwise, we obtain
\[
\rho(s) = a + r \cos \left( \frac{s}{r} \right) \quad 0 \leq s \leq 2\pi r
\]
Then by part a,
\[
A(T) = 2\pi \int_0^{2\pi r} a + r \cos \left( \frac{s}{r} \right) ds = 4\pi^2 ra.
\]

14. a. We have
\[
\langle \nabla f(p), \frac{\partial}{\partial u} \rangle = d_f (\frac{\partial}{\partial u}) = \frac{\partial f}{\partial u} \equiv f_u.
\]
\[
\langle \nabla f(p), \frac{\partial}{\partial v} \rangle = d_f (\frac{\partial}{\partial v}) = \frac{\partial f}{\partial v} \equiv f_v.
\]
Write \( \nabla f(p) = ax_u + by_v \), \( a, b \in \mathbb{R} \). (Of course, \( a \) and \( b \) depend upon \( p \).) Then the above formulas yield
\[
aE + bF = f_u
\]
\[
aF + bG = f_v
\]
Solve for \( a \) and \( b \) to get
\[
a = \frac{f_u G - f_v F}{EG - F^2}
\]
Note that since \( x_u \) and \( x_v \) are linearly independent, \( z_u \wedge z_v \neq 0 \). Thus \( \text{det} F = |z_u \wedge z_v|^2 > 0 \).

b. By the Cauchy-Schwarz inequality,

\[
|f_v(v)| \leq \frac{\langle \text{grad} f(p), v \rangle}{|\text{grad} f(p)|} \leq |\text{grad} f(p)|
\]

whenever \( |v| = 1 \). If \( v = \frac{\text{grad} f(p)}{|\text{grad} f(p)|} \), this maximum is attained. Since equality holds in the Cauchy-Schwarz inequality only when \( v \) and \( \text{grad} f(p) \) are collinear, the maximum of \( f_v(v) \) occurs only when \( v = \frac{\text{grad} f(p)}{|\text{grad} f(p)|} \).

c. Let \( c \in \mathbb{R} \), and let \( C = f^{-1}(c) \subseteq S \). If \( (\text{grad} f)(p) \neq 0 \) for any \( p \in C \), then \( df(p) : T_p S \rightarrow \mathbb{R} \) is onto and thus \( c \) is a regular value. It follows from exercise 28 of 2-4 that \( C \) is a regular curve.

Suppose now that \( \alpha : I \rightarrow C \) is a parametrized curve with \( \alpha(0) = p \). Then \( (f \circ \alpha)(t) = c \forall t \in I \); this implies that

\[
0 = (f \circ \alpha)'(c) = (df)(p)(\alpha'(0)) = \langle \text{grad} f(p), \alpha'(0) \rangle_p.
\]
Since any vector in $T_pC$ is of the form $u'(0)$ for some such $u$, it follows that $\text{grad}f(p)$ is normal to $C$ at $p$.

2-6

1. We prove the contrapositive. Suppose $S$ is an orientable regular surface covered by connected coordinate neighborhoods $U_1$ and $U_2$. Choose an orientation on $S$, and let $\tilde{x}_1: U_1 \rightarrow V_1$ and $\tilde{x}_2: U_2 \rightarrow V_2$ be parametrizations. Then $(d\tilde{x}_i)(q): (\mathbb{R}^2 \rightarrow T_pS)$ is either orientation preserving for all $q \in U_i$ or orientation reversing for all $q \in U_i$, $p = \tilde{x}_i(q)$. Let $\tau(\tilde{x}_i) = 1$ if $(d\tilde{x}_i)(q)$ is orientation preserving, and $\tau(\tilde{x}_i) = -1$ if $(d\tilde{x}_i)(q)$ is orientation reversing. Then the Jacobian of $\tilde{x}_2^{-1} \circ \tilde{x}_1$ at $\tilde{x}_1(p)$ is positive if $\tau(\tilde{x}_1)\tau(\tilde{x}_2) = 1$ and negative if $\tau(\tilde{x}_1)\tau(\tilde{x}_2) = -1$. Thus the sign of the Jacobian of $\tilde{x}_2^{-1} \circ \tilde{x}_1$ must be unchanging on $\tilde{x}_1(U_1 \cap U_2)$.

2. We have $\text{deg}_p: T_pS_1 \rightarrow T_pS_2$ is an isomorphism since $p$ is a local diffeomorphism at $p$. Choose the orientation on $T_pS$, such that $\text{deg}_p$ is orientation preserving. (We assume that we have chosen an
orientation on \( S_2 \). Now choose a coordinate neighborhood \( V \) of \( p \) such that \( \varphi : V \to \varphi(V) \) is a diffeomorphism onto an open set in \( S_2 \). Let \( \pi : U \to V \) be a parametrization. Then \( \varphi \circ \pi : U \to \varphi(V) \) is a parametrization and therefore \( d(\varphi \circ \pi)(q) \) is orientation preserving for all \( q \in U \) or is orientation reversing for all \( q \in U \). Since \( d(\varphi \circ \pi) = dp \circ d\pi \) and we have defined our orientation on each tangent plane of \( S \), so that \( dp \) is orientation preserving, it follows that \( (d\pi)(q) \) is either orientation preserving for all \( q \in U \) or is orientation reversing for all \( q \in U \). Since \( p \in S \), is arbitrary, this shows that \( S_2 \) is orientable.

4. Let \( S \) be a regular, connected, orientable surface, and let \( N : S \to \mathbb{R}^2 \) be an orientation. If \( M : S \to \mathbb{R}^3 \) is another orientation, we have that \( M(p) = \pm N(p) \) for any \( p \in S \). Consider the set 
\[
C = \{ p \in S : M(p) = \pm N(p) \}.
\]
I claim that \( C \) is both open and closed in \( S \). Assuming this, it follows by the connectivity of \( S \) that either \( C = \emptyset \) or \( C = S \). If \( C = S \), we have that \( M = N \); if \( C = \emptyset \), we have
that \( M = -N \). Therefore \( S \) has exactly two orientations.

We now prove the claim. Since \( M \) and \( N \) are continuous, it is immediate that \( C \) is closed. But

\[
S - C = \{ p \in S : M(p) = -N(p) \}
\]

and by the same argument, \( S - C \) is closed. Therefore \( C \) is open as well, completing the proof.