Math 442

Winter 2008

Solutions to Homework 1

1-2.3. If $\alpha''(t)$ is identically 0, then $\alpha'(t)$ is constant. This implies that $\alpha(t) = ct+b$, with c and b constant vectors.

1-2.4. $d(\alpha(t)\cdot v)/dt = \alpha'(t)\cdot v = 0$ for all $t \in I$, by hypothesis. Therefore $\alpha(t)\cdot v = c$, where c is a constant. Since $\alpha(0) \cdot v = 0$, we have that $c = 0$ and therefore that $\alpha(t)$ is orthogonal to v for all $t \in I$.

1-2.5. $|\alpha(t)|$ is constant if and only if $\alpha(t) \cdot \alpha(t)$ is constant. This is the same as requiring that $d(\alpha \cdot \alpha)/dt$ is identically 0. But $d(\alpha \cdot \alpha)/dt = 2\alpha'(t) \cdot \alpha(t)$ and is therefore identically 0 if and only if $\alpha'(t)$ is orthogonal to $\alpha(t)$ for all $t \in I$.

1-3.10 a. Since $d(\alpha(t) \cdot v)/dt = \alpha'(t) \cdot v$, it follows that

$$\int_a^b \alpha'(t) \cdot v \, dt = \alpha(b) \cdot v - \alpha(a) \cdot v = (q-p) \cdot v.$$

Moreover,

$$\int_a^b \alpha'(t) \cdot v \, dt \leq \int_a^b |\alpha'(t) \cdot v| \, dt$$

since $\alpha'(t) \cdot v \leq |\alpha'(t) \cdot v|$. But by the Cauchy-Schwarz inequality, $|\alpha'(t) \cdot v| \leq |\alpha'(t)||v| = |\alpha'(t)|$; hence

$$\int_a^b |\alpha'(t) \cdot v| \, dt \leq \int_a^b |\alpha'(t)| \, dt.$$

This implies that

$$(q-p) \cdot v = \int_a^b \alpha'(t) \cdot v \, dt \leq \int_a^b |\alpha'(t)| \, dt,$$

as desired.

b. This is immediate from part a), since

$$(q-p) \cdot (\frac{q-p}{|q-p|}) = |q-p| = |\alpha(b) - \alpha(a)|.$$

1-4.1 a.

$$\begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix} = 2 - 12 = -10,$$

so $\{(1,3),(4,2)\}$ is not a positive basis.

b.

$$\begin{vmatrix} 1 & 2 & 4 \\ 3 & 3 & 8 \\ 5 & 7 & 3 \end{vmatrix} = 1 \cdot \begin{vmatrix} 3 & 8 \\ 7 & 3 \end{vmatrix} - 2 \cdot \begin{vmatrix} 3 & 8 \\ 5 & 3 \end{vmatrix} + 4 \cdot \begin{vmatrix} 3 & 3 \\ 5 & 7 \end{vmatrix}$$
$$= -47 + 62 + 24 = 39.$$

Therefore, $\{(1,3,5),(2,3,7),(4,8,3)\}$ is a positive basis.

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1-4.12. If there exists a vector u such that $u \wedge v = w$, then w is perpendicular to u and v; in particular, w is perpendicular to v.

Conversely, suppose w is perpendicular to v. If $w = 0$, then $0 \wedge v = w$, so we may take $u = 0$. Otherwise, consider $z = w \wedge v$. Then z, v, w are nonzero and orthogonal; since z, v, $z \wedge v$ are also nonzero and orthogonal, it follows that w is a scalar multiple of $z \wedge v$, say $w = c(z \wedge v)$. We can then take $u = cz$. (You can actually figure out the value of c, if you want.)

Suppose we also have $y \wedge v = w$. Then $(y - u) \wedge v = 0$, so $y - u$ is a scalar multiple of v, say $y - u = bv$. This means that $y = u + bv$.

1-4.13.

$$\begin{aligned}\frac{d}{dt}(u(t) \wedge v(t)) &= u'(t) \wedge v(t) + u(t) \wedge v'(t) \\ &= (au(t) + bv(t)) \wedge v(t) + u(t) \wedge (cu(t) - av(t)) \\ &= au(t) \wedge v(t) - u(t) \wedge av(t),\end{aligned}$$

since $bv(t) \wedge v(t) = 0 = u(t) \wedge cu(t)$. But

$$au(t) \wedge v(t) - u(t) \wedge av(t) = a(u(t) \wedge v(t)) - a(u(t) \wedge v(t)) = 0.$$

This therefore implies that $u(t) \wedge v(t)$ is constant.