4. Let $S = \{(x, y, z) : xy = z^2\}$. Then $S$ is parametrized by $\mathbf{x} : \mathbb{R}^2 \to S$ defined by $\mathbf{x}(u, v) = (u, v, uv)$.

With this coordinate system we have

\[
\begin{align*}
E(u, v) &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle = 1 + v^2 \\
F(u, v) &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle = uv \\
G(u, v) &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle = 1 + u^2
\end{align*}
\]

and, from Example 5 of 3–3,

\[
\begin{align*}
e(u, v) &= 0 \\
f(u, v) &= \frac{1}{(1 + u^2 + v^2)^{\frac{1}{2}}} \\
g(u, v) &= 0
\end{align*}
\]

Now if $\alpha : I \to S$ is a parametrization of a regular curve in $S$, which we write as

\[\alpha(t) = \mathbf{x}(u(t), v(t))\]

then it follows from Equation (7) on p. 160 that $\alpha(t)$ is an asymptotic curve if and only if

\[
\frac{u'(t) v'(t)}{\left[1 + (u'(t))^2 + (v'(t))^2\right]^\frac{1}{2}} = 0
\]
But this is true if and only if \( u'(t) v'(t) = 0 \) for all \( t \in I \), and this implies that either \( u'(t) = 0 \) for all \( t \) or \( v'(t) = 0 \) for all \( t \). (Why? Since \( a \) is regular, either \( u'(t) \neq 0 \) or \( v'(t) \neq 0 \), for any \( t \in I \). Thus \( \{ t \in I : u'(t) = 0 \} \) and \( \{ t \in I : v'(t) = 0 \} \) are disjoint closed sets whose union is \( I \). Since \( I \) is connected, one of these sets must be all of \( I \).)

From this it follows that the asymptotic curves are (arcs of) the curves \( v = \text{constant} \) or \( u = \text{constant} \).

As for the lines of curvature, \( \alpha(t) = x(u(t), v(t)) \) must satisfy

\[
\frac{1 + \dot{u}^2}{(1 + u^2 + v^2)^{3/2}} (u')^2 = \frac{1 + u^2}{(1 + u^2 + v^2)^{3/2}} (v')^2 = 0
\]

(do Carmo p. 161). Thus

\[
\frac{(u')^2}{1 + u^2} = \frac{(v')^2}{1 + v^2},
\]

so either

\[
\frac{u'(t)}{(1 + u^2)^{1/2}} = \frac{v'(t)}{(1 + v^2)^{1/2}} \text{ for all } t \in I.
\]
or
\[ \frac{u'(t)}{(1 + u^2)^{1/2}} = -\frac{v'(t)}{(1 + v^2)^{1/2}} \quad \text{for all } t \in \mathbb{R}. \]

(Why?) Hence
\[ \int \frac{u'(t)}{(1 + u^2)^{1/2}} \, dt = \pm \int \frac{v'(t)}{(1 + v^2)^{1/2}} \, dt. \]
\[ \int \frac{du}{(1 + u^2)^{1/2}} = \pm \int \frac{dv}{(1 + v^2)^{1/2}}. \]

Now recall that
\[ \int \frac{dx}{(1 + x^2)^{1/2}} = (\sinh^{-1} x) + C, \]

so we obtain
\[ (\sinh)^{-1} u + (\sinh)^{-1} v = C \]

or
\[ (\sinh)^{-1} u - (\sinh)^{-1} v = C \]
as the lines of curvature.

G. a. As in the hint, we take \( r \) to be the z-axis, and a normal to \( r \) as the x-axis. We also restrict
attention to the part of \( C \) lying in the first quadrant, so that \( C \) is the graph of a function \( z \) of \( x \).

Now the tangent line to \( C \) at \((x, z(x))\) intersects the \( z\)-axis at the point \((0, -z(x)x + z(x))\).

We want the line segment from \((x, z(x))\) to \((0, -z(x)x + z(x))\) to have length 1; i.e.

\[
2'(x) x^2 + x^2 = 1
\]

\[ z'(x) = -\frac{\sqrt{1-x^2}}{x} \]

Hence

\[
z(x) = -\int \frac{\sqrt{1-x^2}}{x} \, dx = -\int \frac{\cos^2\theta}{\sin\theta} \, d\theta \quad \theta = \sin^{-1} x, \quad 0 < \theta < \frac{\pi}{2}
\]

\[ = -\int \csc\theta - \sin\theta \, d\theta \]

\[ = -\ln(\csc\theta - \cot\theta) - \cos\theta + C \]

\[ = -\ln \left( \frac{1}{x} - \frac{\sqrt{1-x^2}}{x} \right) - \sqrt{1-x^2} + C \]
Since \( z(1) = 0 \), we get \( C = 0 \), so

\[
z(x) = -\ln\left(\frac{1}{x} - \frac{1-x^2}{x}\right) - \sqrt{1-x^2}
\]

C. The curve \( C \) between \( x = 0 \) and \( x = 1 \) is parametrized by

\[
\begin{align*}
\varphi_0(x) &= x \\
\varphi_0(x) &= z(x) = -\ln\left(\frac{1}{x} - \frac{1-x^2}{x}\right) - \sqrt{1-x^2}
\end{align*}
\]

\[0 < x < 1.\] Since

\[
\varphi_0'(x)^2 + \varphi_0'(x)^2 = 1 + \frac{1-x^2}{x^2} = \frac{1}{x^2},
\]

it follows that

\[
\begin{align*}
\varphi(v) &= x^v \\
\varphi'(v) &= \varphi_0(2^v)
\end{align*}
\]

is a parametrization of \( C \) by arc length.

By example 4 on pp. 16-1-2 of do Carmo, it follows that the Gaussian curvature of the pseudosphere is given by

\[
K(x, v) = -\frac{\varphi''(v)}{(\varphi(v))^3}.
\]

(By symmetry, the curvature is independent of the parameter \( u \).)
But
\[- \frac{\psi''(v)}{\psi(v)} = - \frac{d}{x} = -1,\]
completing the proof.

7. 2. From the equation \( K(u,v) = -\frac{\psi''(v)}{\psi(v)} \), it follows that \( K \equiv 0 \) implies \( \psi''(v) = 0 \), which in turn implies that \( \psi(v) = av + b \) for some constants \( a \) and \( b \).

Then \( \psi'(v) = \pm \sqrt{1 - \psi'(v)^2} = \pm \sqrt{1 - a^2} \) so that \( \psi'(v) \) is constant as well. We then get that \( \psi(v) = cv + d \) for some constants \( c \) and \( d \) with \( a^2 + c^2 = 1 \).

If \( a = 0 \), the function \( (\psi(v), \psi'(v)) \) is a parametrization of the line \( x = b \), so the surface of revolution is a right circular cylinder. If \( a \neq 0 \), the function \( (\psi(v), \psi'(v)) \) is a parametrization of the line \( z = c(\frac{x-b}{a}) + d \), so the surface of revolution is a right circular cone if \( c \neq 0 \) and a plane if \( c = 0 \).

13. We prove a more general result:

Proposition: Let \( F: \mathbb{R}^2 \to \mathbb{R}^3 \) be a diffeomorphism, and let \( S \subset \mathbb{R}^3 \) be a regular surface. Then \( F(S) \) is also a regular surface.
Proof: Let $\overline{p} \in \overline{S}$, $p = F^{-1}(\overline{p}) \in S$. Since $S$ is a regular surface, there exists a differentiable map $x: U \to V$, where $U$ is open in $\mathbb{R}^2$ and $V$ is a neighborhood of $p$ in $S$ such that

i. $x$ is a homeomorphism

ii. $(dx)(g): \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one for each $g \in U$.

Now consider $F \circ x$. $F \circ x$ is differentiable since both $F$ and $x$ are; moreover, $F \circ x: U \to F(V)$ is a homeomorphism and $F(V)$ is a neighborhood of $\overline{p}$ in $\overline{S}$, since $F: S \to \overline{S}$ is a homeomorphism. Finally,

$$d(F \circ x)(g) = (dF)(x(g)) \cdot dx(g)$$

is one-to-one for each $g \in U$ because $(dF)(x(p))$ is invertible. We have therefore found a parametrization of a neighborhood of each point of $\overline{S}$ so that $\overline{S}$ is a regular surface.

Now let $F$ be defined by $F(p) = cp$, $c$ a positive constant. If $N: S \to S^2$ is a unit normal vector field on $S$, then $\overline{N}: \overline{S} \to S^2$ defined by $\overline{N}(\overline{p}) = N(p)$ is a unit normal vector field on $\overline{S}$. This can be done because $T_p S = T_{\overline{p}} \overline{S}$ (Why?). Since $\overline{N}$ is
the composition
\[ \mathbf{S} \xrightarrow{F} \mathbf{S} \xrightarrow{N} \mathbf{S}^2. \]

it follows that
\[ (dN)(\overline{F}) : T_{\overline{F}} \mathbf{S} \rightarrow T_{\overline{F}} \mathbf{S} \]
is just the composition
\[ (dN)(\overline{F}) \circ (dF^{-1})(\overline{F}) : T_{\overline{F}} \mathbf{S} \rightarrow T_{\overline{F}} \mathbf{S} = T_{\overline{F}} \mathbf{S}. \]

But \((dF^{-1})(\overline{F}) : T_{\overline{F}} \mathbf{S} \rightarrow T_{\overline{F}} \mathbf{S}\) sends \(v\) to \(\overline{v}\), so
\[ dN(\overline{F})(v) = \frac{1}{c} (dN(\overline{F}))(v). \]

Therefore, the eigenvalues of \((dN)(\overline{F})\) are \(\frac{1}{c}\) times the eigenvalues of \(dN(\overline{F})\), from which we conclude that
\[ \overline{K}(\overline{F}) = \frac{1}{c} K(\overline{F}) \]
\[ \overline{H}(\overline{F}) = \frac{1}{c} H(\overline{F}). \]

Note: If \(F : \mathbb{R}^2 \rightarrow \mathbb{R}^2\) is an arbitrary diffeomorphism, it is not the case that \(T_{\overline{F}}(\mathbf{S}) = T_{F}(\mathbf{S})\). We thus cannot expect such a nice relation between \(\overline{K}\) and \(K\) or \(\overline{H}\) and \(H\).
16. Let \( S \) be a compact regular surface in \( \mathbb{R}^3 \) and consider the differentiable function \( f: S \to \mathbb{R} \) defined by \( f(x, y, z) = x^2 + y^2 + z^2 \). Since \( S \) is compact, there exists \( p = (x_0, y_0, z_0) \) for which \( f(p) \) is a maximum. Clearly, \( p \) is not at the origin. I claim that the orientation \( N: S \to S^2 \) can be chosen so that \( \nabla f(p) \cdot v < 0 \) for all \( v \in T_p S \) with \( |v| = 1 \). Assuming the claim, it follows that \( \lambda_1 < 0 \) and \( \lambda_2 < 0 \) (since if \( (dN_p)(v) = -\lambda_1 v_1 \) and \( |v_1| = 1 \), we have \( \nabla f(v_1) = -\lambda_1 v_1 \)), so that \( p \) is an elliptic point.

Proof of claim: Let \( \alpha: I \to S \) be a regular curve parametrized by arc length with \( \alpha(0) = p \). Then the function \( \alpha\cdot \alpha' \) attains its maximum at \( t = 0 \), so

1) \( (\alpha \cdot \alpha')(0) = 0 \)

2) \( (\alpha \cdot \alpha')(0) \leq 0 \)

Equation 1) implies that \( p \cdot \alpha'(0) = \alpha(0) \cdot \alpha'(0) = 0 \); since \( \alpha \) was arbitrary, this proves that \( p \) is normal to \( T_p S \). We may thus choose \( N: S \to S^2 \) so that \( N(p) = \frac{p}{|p|} \). Now equation 2) implies that
\[ 0 \geq a'(0) \cdot a'(0) + a(0) \cdot a''(0) > |p| N(p) \cdot a''(0) \]
\[ \geq |p| K_n(p) \]
\[ \geq |p| \Pi_p(a'(0)). \]

Therefore, \( \Pi_p(a'(0)) < 0 \). Since \( a \) was arbitrary, this completes the proof.

3.5

3.5a Let \( S \) be a regular surface without umbilical points. If \( S \) is a minimal surface, then \( H \equiv 0 \) and \( S \) has no planar points (since any planar point is an umbilical point). It then follows from exercise 17 of 3.2 that
\[ \langle dN_p(w_1), dN_p(w_2) \rangle = -K(p) \langle w_1, w_2 \rangle \]
for all \( p \in S \) and \( w_1, w_2 \in T_pS \).

Conversely, suppose \( S \) is a regular surface without umbilical points and that
\[ \langle dN_p(w_1), dN_p(w_2) \rangle = \lambda(p) \langle w_1, w_2 \rangle \]
for all \( p \in S \) and \( w_1, w_2 \in T_pS \). Let \( e_1 \) be a \( \lambda \)-eigenvector with eigenvalue \( \lambda_1 \), and let \( e_2 \) be a \( \lambda \)-eigenvector with eigenvalue \( \lambda_2 \). Then
\[ \lambda(p) = \lambda(p) \langle e_1, e_2 \rangle = \langle dN_p(e_1), dN_p(e_2) \rangle = \lambda_1^2, \]
and
\[ \lambda(p) = \lambda(p) (e_1, e_2) = \langle dN_p (e_1), dN_p (e_2) \rangle = \delta_{12} \]

Since \( S \) has no umbilical points, this implies that \( \delta_{13} = \delta_{22} \).

Therefore, \( \mu = 0 \) and \( S \) is a minimal surface.

b. If \( \sigma \) is isothermal, then there exists a function \( \zeta: \mathbb{U} \to (0, \infty) \) such that

\[ \langle d\xi_{x(u,v)}(v_1), d\xi_{x(u,v)}(v_2) \rangle = \zeta(u,v) \langle v_1, v_2 \rangle \]

for all \((u,v) \in \mathbb{U}\) and \(v_1, v_2 \in \mathbb{R}^2\). Also, \( d(N^{-1})_{\theta} = (dN_{\theta})^{-1}\),

where \( N(\theta) = \theta \); it then follows from part a) that

\[ \lambda(p) \langle d(N^{-1})_{\theta} z_1, d(N^{-1})_{\theta} z_2 \rangle = \langle dN_{\theta} (dN^{-1})_{\theta} (z_1), dN_{\theta} (dN^{-1})_{\theta} (z_2) \rangle \]

for all \( z_1, z_2 \in T_{\theta} S^2 = T_{\theta} S \). Therefore, if \( v_1, v_2 \in \mathbb{R}^2 \) and \( \sigma(u,v) = \theta \), \( N^{-1}(\theta) = \rho \), we have that

\[ \frac{c(u,v)}{\lambda(p)} \langle v_1, v_2 \rangle = \frac{1}{\lambda(p)} \langle d\xi_{x(u,v)}(v_1), d\xi_{x(u,v)}(v_2) \rangle \]

\[ = \langle d(N^{-1})_{\theta} (d\xi_{x(u,v)}(v_1)), d(N^{-1})_{\theta} (d\xi_{x(u,v)}(v_2)) \rangle \]

\[ = \langle d(N^{-1})_{\theta} (v_1), d(N^{-1})_{\theta} (v_2) \rangle. \]

This proves that \( N^{-1}\theta \) is isothermal.