do Carmo

4-4
1. a. Let $\alpha : I \rightarrow S$ be a (local) parametrization of $C$ by arc length, and let $N$ be a local orientation of $S$. Since $\alpha$ is a geodesic, $\alpha''(t)$ is a scalar multiple of $N(\alpha(t))$, say $\alpha''(t) = c(t) N(\alpha(t))$ for each $t$. But $C$ is also a line of curvature; thus

$$\frac{d}{dt} (N \cdot \alpha)(t) = dN_{\alpha(t)} (\alpha'(t)) = \lambda(t) \alpha'(t)$$

for all $t$. I now claim that the unit vector $\alpha'(t) \wedge N(\alpha(t))$ is constant for $t \in I$. Indeed,

$$\begin{align*}
\frac{d}{dt} (\alpha'(t) \wedge N(\alpha(t))) &= \alpha''(t) \wedge N(\alpha(t)) + \alpha'(t) \wedge dN_{\alpha(t)} \alpha'(t) \\
&= c(t) N(\alpha(t)) \wedge N(\alpha(t)) + \alpha'(t) \wedge \lambda(t) \alpha'(t) \\
&= 0.
\end{align*}$$

Moreover, fix $t_0 \in I$ and consider

$$\langle \alpha(t) - \alpha(t_0), \alpha'(t) \wedge N(\alpha(t)) \rangle.$$ 

We have

$$\frac{d}{dt} \langle \alpha(t) - \alpha(t_0), \alpha'(t) \wedge N(\alpha(t)) \rangle = \langle \alpha'(t), \alpha'(t) \wedge N(\alpha(t)) \rangle = 0,$$

and thus $\alpha(t) - \alpha(t_0)$ lies in the plane perpendicular to $\alpha'(t_0) \wedge N(\alpha(t_0))$. Hence $\alpha$ is a plane curve,
and it's not hard to prove, using the connectedness of C, that C is a plane curve as well.

6. We assume \( \alpha: \mathbb{R} \to S \) is a plane curve parametrized by arclength with \( \alpha''(t) \) never zero. Then \( \alpha'(t) \cdot \frac{\alpha''(t)}{|\alpha''(t)|} \) is a unit vector perpendicular to the plane spanned by \( \alpha'(t) \) and \( \alpha''(t) \). Since \( \alpha \) is a plane curve, this plane is independent of \( t \); thus \( \alpha'(t) \cdot \frac{\alpha''(t)}{|\alpha''(t)|} \) is constant for all \( t \). Now suppose \( \alpha \) is also a geodesic. Then \( \frac{\alpha''(t)}{|\alpha''(t)|} = N(\alpha(t)) \) for some choice of local orientation on \( S \), and

\[
0 = \frac{d}{dt} \left( \alpha'(t) \cdot \frac{\alpha''(t)}{|\alpha''(t)|} \right) = \alpha'(t) \cdot \frac{d}{dt} \left( \frac{\alpha''(t)}{|\alpha''(t)|} \right) = \alpha'(t) \cdot \frac{\alpha''(t)}{|\alpha''(t)|} N(\alpha(t))
\]

This implies that \( dN_{\alpha(t)}(\alpha'(t)) \) is a scalar multiple of \( \alpha'(t) \); in other words, \( \alpha \) is a line of curvature of \( S \).

7. A circle of latitude of the sphere, which is not the equator.

4. First observe that

\[
\frac{d}{dt} \frac{du}{dt} = \frac{d}{dt} + u(t).
\]
where \( u(t) \) is perpendicular to \( T_{a(t)}S \). Similarly,

\[
\frac{d}{dt} u(t) = \frac{Du}{dt} + r(t),
\]

where \( r(t) \) is perpendicular to \( T_{a(t)}S \). Then

\[
\frac{d}{dt} \langle v(t), w(t) \rangle = \langle \frac{dv}{dt}, w(t) \rangle + \langle v(t), \frac{dw}{dt} \rangle 
= \langle \frac{dv}{dt}, w(t) \rangle + \langle v(t), \frac{dw}{dt} \rangle
\]

Since \( \langle u(t), w(t) \rangle = 0 = \langle v(t), r(t) \rangle \).

5 a. By example 5 of do Carmo, a parallel is a geodesic if and only if the line tangent to the generating curve at the point generating the parallel is vertical. This only happens with the maximum and minimum parallels.

![Diagram](attachment:diagram.png)

Picture.

5 b. A regular parametrized curve \( \alpha \) is an asymptotic curve if and only if \( \alpha''(t) \in T_{a(t)}S \) for all \( t \). This is only the case with the upper parallel.
c. All parallels on a surface of revolution are lines of curvature (see example 4 in 3.3 of do Carmo).

11. Proposition. Let $S_1$ and $S_2$ be regular oriented surfaces and $\varphi: S_1 \to S_2$ an orientation preserving isometry. Then

$$\left[ \frac{D\varphi}{dt} \right] = \frac{D(d\varphi(w))}{dt},$$

where $(d\varphi)(w)(t) = d\varphi_{\varphi(t)}(w(t))$ and is thus a field of unit vectors along the parametrized curve $\varphi: I \to S_2$.

Proof. We proved in class that, for each $s \in I$,

$$d\varphi_{\varphi(s)} \left( \frac{Dw}{dt} (s) \right) = \frac{D(d\varphi(w))}{dt} (s).$$

It therefore suffices, by Definition 9 in do Carmo, to prove that

$$d\varphi_{\varphi(s)} (N_1 \wedge w(s)) = N_2 \wedge d\varphi_{\varphi(s)}(w(s)),$$

where $N_1, N_2$ are the positively oriented unit normals to $T_{\varphi(s)} S_1$ and $T_{\varphi(s)} S_2$ respectively. Let $z_1, z_2$ be a positively oriented orthonormal basis of $T_{\varphi(s)} S_1$; since $\varphi$ is an orientation preserving isometry,
\( d\mathbf{q}_{u(v)}(z_1), d\mathbf{q}_{u(v)}(z_2) \) is a positively oriented orthonormal basis of \( T_{\mathbf{q}(u(v))} S_2 \). Moreover, \( z_1, z_2, N_1 \) and \( d\mathbf{q}_{w(v)}(z_1), d\mathbf{q}_{w(v)}(z_2), N_2 \) are positively oriented orthonormal bases of \( \mathbb{R}^3 \). We may thus extend \( (d\mathbf{q}_{u(v)}) \) to an orthogonal linear transformation \( T: \mathbb{R}^3 \to \mathbb{R}^3 \) with positive determinant by defining \( T(N_1) = N_2 \). It then follows from exercise 6b of 1-5 that

\[
d\mathbf{q}_{w(v)}(N_1 \wedge w(v)) = T(N_1 \wedge w(v)) = T(N_1) \wedge T(w(v)) = N_2 \wedge d\mathbf{q}_{w(v)}(w(v)).
\]

This completes the proof.

12. Let \( p \in S \). By Corollary 1 of 3-4, we may choose a parametrization \( x: U \to S, x(p) = p \), of a neighborhood of \( p \) such that the coordinate curves \( x(t, v) \) and \( x(u, t) \) are curves in the two families respectively. Since the families are orthogonal, we have that \( \langle x_v, x_u \rangle = 0 \), so \( x \) is an orthogonal parametrization.

Now, for each fixed \( v \), the trace of \( x(t, v) \) is a geodesic, since each family consists entirely of geodesics.
Thus there exists a strictly monotone function \( t = \psi(s) \) such that \( y(s) = y (\psi(s), v) \) is a parametrized geodesic. (Of course, \( \psi \) depends upon \( v \).) Then \( \frac{d^2 y}{ds^2} \) is orthogonal to \( T_{\psi(s)} S \); in particular, 
\[
\left\langle \frac{d^2 y}{ds^2}, x_v \right\rangle = 0.
\]

But
\[
\frac{d^2 y}{ds^2} = \frac{d}{ds} \left[ \psi'(s) x_u (\psi(s), v) \right] = \psi''(s) x_u (\psi(s), v) + \psi'(s)^2 x_{uu} (\psi(s), v).
\]

Since \( \left\langle x_u, x_v \right\rangle = 0 \) and \( \psi'(s) \) is never 0, it follows that \( \left\langle x_{uu}, x_v \right\rangle = 0 \). Now
\[
\left\langle x_{uu}, x_v \right\rangle = F_u - \frac{1}{2} E_v = -\frac{1}{2} E_v
\]
as \( F \equiv 0 \).

Similarly, we know that, for each fixed \( u \), the trace of \( x(u,t) \) is a geodesic, and this implies that \( G_u \equiv 0 \).

Now use the formula (Exercise 1 of 4-3)
\[
K = -\frac{1}{2} \text{Tr} \left\{ \left( \frac{E_u}{\text{Tr} E} \right)_v + \left( \frac{G_u}{\text{Tr} E} \right)_u \right\}.
\]
valid for any orthogonal parametrization $z$, to conclude that $K = 0$ on $\Sigma(U)$. 