1. Consider the ellipsoid $S$ given by

$$x^2 + 2y^2 + 4z^2 = 1.$$ 

a. Prove that, if $(x, y, z) \in S$, the unit normal vector $N(x, y, z)$ is a scalar multiple of the vector $U(x, y, z) = (x, 2y, 4z)$. (You may use any general theorem from this course, but you must state it precisely.)

b. Suppose that $p_0 = (x_0, y_0, z_0)$ is an umbilical point; that is, the linear operator $dN_{p_0}$ has only one eigenvalue. Prove that, if $\alpha : I \to S$ is any parametrized curve with $\alpha(0) = p_0$, then

$$\left( \frac{dU}{dt}(0) \land \frac{d\alpha}{dt}(0), U(0) \right)_{p_0} = 0.$$ 

(By abuse of notation, we are here writing $U(t) = U(x(t), y(t), z(t))$, where $\alpha(t) = (x(t), y(t), z(t))$.) Hint: Write $U(t) = f(t)N(t)$ and use the product rule to compute $\frac{dU}{dt}(0)$.

Let $f(x, y, z) = x^2 + 2y^2 + 4z^2$. Since 1 is a regular value of $f$, the normal vector to $S$ at $(x, y, z)$ is given by a scalar multiple of $(\frac{\partial f}{\partial x}(x, y, z), \frac{\partial f}{\partial y}(x, y, z), \frac{\partial f}{\partial z}(x, y, z)) = (2x, 4y, 8z)$. This is just a scalar multiple of $(x, 2y, 4z)$.

b. $\frac{dU}{dt}(0) = f'(0)N(0) + f(0)N'(0)$

$$= f'(0)N(0) + f(0) \frac{dN_{p_0}(\alpha'(0))}{dt}$$

$$= f'(0)N(0) + f(0) \cdot k \alpha'(0)$$

where $k$ is the unique eigenvalue of $dN_{p_0}$. (Note that every vector in $T_{p_0}S$ is an eigenvector.) But

$N(0) \land \alpha'(0)$ is orthogonal to $N(0)$, so $\langle f'(0)N(0) \land \alpha'(0), U(0) \rangle = 0$ and $\alpha'(0) \land \alpha'(0) = 0$, so $\langle f(0)N(0) \land \alpha'(0), U(0) \rangle = 0$.

Thus $\langle U'(0) \land \alpha'(0), U(0) \rangle = 0$. 

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(6) c. With the hypotheses of part b), assume in addition that \( y_0 = 0 \) and \( z_0 \neq 0 \). Choose \( \alpha \) so that \( x'(0) = 1 \) and \( y'(0) = 1 \). Apply the equation in part b) to prove that \( z_0^2 = \frac{1}{12} \).

Write \( x'(0) = (1, 1, z') \). We then have

\[
\langle (1, 2, 4z'), (1, 1, z'), (x_0, 0, 4z_0) \rangle = 0.
\]

This implies

\[
\langle (-2z', 3z', -1), (x_0, 0, 4z_0) \rangle = 0,
\]

so that

\[(*) \quad -2z'x_0 = 4z_0.\]

But \( x'(0) \in T_{(x_0, 0, z_0)} S \); thus by part a),

\[
x_0 + 4z_0z' = 0;
\]

i.e.

\[
z' = -\frac{x_0}{4z_0}.
\]

Plugging into \((*)\) yields

\[
x_0^2 = 8z_0^2.
\]

Since \( x_0^2 + 4z_0^2 = 1 \), we obtain \( z_0^2 = \frac{1}{12} \).