Math 443

Spring 2008

Homework 4

Read Section 4-5 (up to page 279) in do Carmo. Do exercise 1 of 4-5 as well as the following supplementary exercise.

Supplementary Exercise: The goal of this exercise is to prove that if S is a compact connected surface in $\mathbb{R}^3$ with $K(p) > 0$ for all p, then S is orientable and the Gauss map $N : S \to S^2$ is a diffeomorphism. A quicker proof of this can be given if one knows about covering spaces; however, we will prove it without using covering spaces.

a. Prove that S is orientable. (It is a general fact that all compact regular surfaces in $\mathbb{R}^3$ are orientable; you are asked to prove this in this special case.) Hint: Use Proposition 1 of 3-3.

b. Prove that N is a local diffeomorphism; that is, for each $p \in M$, there exist open neighborhoods U and V of p and $N(p)$ respectively such that $N : U \to V$ is a diffeomorphism. Therefore, to prove that N itself is a diffeomorphism, it suffices to prove that N is bijective.

c. Prove that N is onto by showing that $N(S)$ is both open and closed in S^2.

In parts d–g, you will show that N is one-to-one by assuming that $N(p) = N(q)$ for some $p, q \in S$, $p \neq q$, and deriving a contradiction.

d. Prove that there is an open neighborhood U of q such that $N(S - \overline{U}) = S^2$.

e. Suppose V is a coordinate neighborhood in $S - \overline{U}$ on which $N : V \to N(V)$ is a diffeomorphism. Prove that $\int_V K\, d\sigma = \int_{N(V)} d\sigma$.

f. Use parts (d) and (e) to prove that

$$\int_{S-\overline{U}} K\, d\sigma \geq \int_{S^2} d\sigma = 4\pi.$$

(Your argument need not be completely rigorous here.)

g. Use the Gauss-Bonnet theorem to show that $\int_{S-\overline{U}} K\, d\sigma \geq 4\pi$ leads to a contradiction.

This assignment is due Friday, June 6.