(1) Let V denote the set of infinitely differentiable functions on the interval $[0, 1]$. It is a vector space under the oprations of addition of functions and multiplication by a real number.

Let

$$W = \{f \in V \,:\, f'' + pf' + qf = 0\}$$

for $p \in V$ and $q \in V$.

(a) Prove that W is a subspace of V.

(b) What does the existence and uniqueness theorem for ordinary differential equations tell you about the dimension of W?

(2) Let $M(n)$ be the set of $n \times n$ matrices.

(a) Show that $M(n)$ is a vector space, where addition is addition of matrics and multiplication by scalars is defined as on page 45 of Lang.

(b) Show that the subset $S(n) \subset M(n)$ of symmetric matrices is a subspace. What is it's dimension?

(c) Show that the subset $A(n) \subset M(n)$ of skew-symmetric matrices is a subspace. What is it's dimension?

(d) Show that $M(n) = S(n) + A(n)$ and $S(n) \cap A(n) = O$.

(3) In problem (1) above you showed that the space of infinitely differentiable functions on the unit interval is a vector space. Show that V is not finite dimensional. **Hint:** *Consider the functions* $f_k(x) = x^k$, $k = 0, 1, 2, 3, \ldots$.