(1) (a) Show that the complex numbers $\mathbb{C}$ may be viewed as a 2-dimensional vector space.

(b) Let $M(2)$ denote the space of 2×2 matrices, and let $L : \mathbb{C} \to M(3)$ be the map defined by

$$L(x + iy) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

Verify that L is a linear map. What is the rank of L?

(c) Show that L satisfies the identity $L(z_1 z_2) = L(z_1)L(z_2)$ for all $z_1, z_2 \in \mathbb{C}$.

(2) Let A be a $n \times n$ matrix and let A_i denote the i-th row of A. We say that A is an *orthogonal* matrix if it satisfies the identities

$$A_i \cdot A_i = 1 \textbf{ and } A_i \cdot A_j = 0 \text{ for all } i \neq j.$$

The set of all $n \times n$ orthogonal matrices is called the *orthogonal group* and is denoted by $O(n)$.

(a) Show that $A^{-1} = A^t$ for all $A \in O(n)$.

(b) Show that $A^t \in O(n)$ for all $A \in O(n)$.

(c) Show that $AB \in O(n)$ for all $A, B \in O(n)$.

(3) Let $L : \mathbb{R}^3 \to M(3)$ be the linear map defined by

$$L(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})) = \begin{pmatrix} 0 & z & y \\ -z & 0 & x \\ -y & -x & 0 \end{pmatrix}$$

Observe that the image of L is the space $A(3)$ of 3×3, skew-symmetric matrices. Show that

$$L(\mathbf{u} \times \mathbf{v}) = [L(\mathbf{u}), L(\mathbf{v})]$$

where $[A, B]$ denotes the Lie bracket of A and B, i.e. $[A, B] = AB - BA$.