(1) Let V be the vector space of continuous function on the interval $[0, \pi]$, that vanish at 0 and π; and let $\langle\, , \rangle$ be the scalar product defined by

$$\langle f, g\rangle = \int_0^\pi f(x)g(x)\,dx\,.$$

Let $g_k(x) = \sin(kx)$, for $k = 1, 2, 3, \ldots$, and let $W_n \subset V$ be the subspace generated by the set $\{g_k\,:\, k = 1, 2, \ldots, n\}$.

(a) Show that $\{g_k\,:\, k = 1, 2, \ldots\}$ is an orthogonal set.

(b) Let $f(x) = x(\pi - x)$. Let f_n denote the orthogonal projection of f onto W_n. Show that

$$f_{2n+1}(x) = \frac{8}{\pi}\sum_{k=0}^{n}\frac{\sin((2k+1)x)}{(2k+1)^3}\,.$$

(2) Let V be the vector space of continuous functions on the closed interval $[-1, 1]$, with scalar product defined by

$$\langle f, g\rangle = \int_{-1}^{+1} f(x)g(x)\,dx\,.$$

(a) Apply the Gram-Schmidt orthogonalization process to the set $\{1, x, x^2, x^3\}$ to obtain an orthogonal set of four polynomials, $\{p_0(x), p_1(x), p_2(x), p_3(x)\}$. (Note that $p_0(x) = 1$ and $p_1(x) = x$.)

(b) Verify that p_k is a solution of the differential equation

$$(1 - x^2)y'' - 2xy' + \lambda y = 0, \text{ with } \lambda = k(k+1)\,.$$

Remark: Applying Gram-Schmidt to the set $\{1.x.x^2, x^3, \ldots\}$ yields an orthogonal set $\{p_k(x)\,:\, k = 0, 1, 2, \ldots\}$ of polynomials, which after multiplication by constnts are called *Legendre polnomials*. Moreover, $p_k(x)$ is a solution of the *Legendre equation*

$$(1 - x^2)y'' - 2xy' + \lambda y = 0 \text{ with } \lambda = k(k+1)\,.$$