Math 307J Autumn 2015
Lecture 14: The Damped Harmonic Oscillator

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- Reading: §3.7.
- Homework: §3.7 #1,2,3,4,6,7,13,16,27.
The system we want to study consists of a mass $m$ attached to a spring, and also attached to a damping mechanism. Later, we will include an external force $f(t)$.

There are three forces acting on the mass:

- $F_{\text{damping}} = -c \frac{dx}{dt}$, the damping force, where $c$ is the damping constant,
- $F_{\text{spring}} = -kx$, the spring force, where $k$ is the spring constant.
- $F_{\text{external}} = f(t)$

The law of motion of the system is thus

$$m \frac{d^2x}{dt^2} = -c \frac{dx}{dt} - kx + f(t)$$

or

$$m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = f(t)$$
Simple case: \( m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + k x = 0 \)

There are four cases, based on the number and type of the roots

\[
    r_1, r_2 = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}
\]

of the characteristic polynomial \( mr^2 + cr + k \):

- \( c^2 - 4mk > 0 \) (Over-damped) Roots are real and negative. General solution:
  \[
  x = A e^{-\left(\frac{c}{2m} + \frac{\sqrt{c^2 - 4mk}}{2m}\right)t} + B e^{-\left(\frac{c}{2m} - \frac{\sqrt{c^2 - 4mk}}{2m}\right)t}.
  \]

- \( c^2 - 4mk = 0 \) (Critically-damped) A single double root.
  \[
  x = (A + Bt) e^{-\frac{c}{2m} t}
  \]

- \( c^2 - 4mk < 0 \) (Under-damped) Roots are complex conjugates of one another.
  \[
  x = A e^{-\frac{c}{2m} t} \cos(\omega t - \varphi), \quad \text{where} \quad \omega = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}.
  \]

- \( c = 0 \) (Un-damped) Roots are pure imaginary.
  \[
  x = A \cos(\omega_0 t - \varphi), \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}} \text{ the (natural frequency).}
  \]
The Damped Harmonic Oscillator

No external force \( f(t) = 0 \)

The Phase Plane

These 4 cases are illustrated in next figure:

<table>
<thead>
<tr>
<th>Case</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Over-damped:</td>
<td>( u = A e^{-\left(\frac{c}{2m} + \frac{\sqrt{c^2 - 4mk}}{2m}\right)t} + B e^{-\left(\frac{c}{2m} - \frac{\sqrt{c^2 - 4mk}}{2m}\right)t} )</td>
</tr>
<tr>
<td>Critically damped:</td>
<td>( u = (A + Bt) e^{-\frac{c}{2m}t} )</td>
</tr>
<tr>
<td>Under-damped:</td>
<td>( u = A e^{-\frac{c}{2m}t} \cos(\omega t - \varphi) )</td>
</tr>
<tr>
<td>Undamped:</td>
<td>( u = A \cos(\omega_0 t - \varphi) )</td>
</tr>
</tbody>
</table>

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The Phase Plane of \( x'' + x = 0 \)

It is equivalent to the system

\[
x' = y \quad \text{and} \quad y' = -x.
\]

Vector field: \( \vec{F} = y\vec{i} - x\vec{j} \).

The general solution of \( x'' + x = 0 \) is \( x = A \cos(t - \varphi) \).

Set \( y = x' \) gives the parametric curve

\[
x = A \cos(t - \varphi) \quad y = -A \sin(t - \varphi),
\]

the equation for a circle of radius \( A \) about the origin.

The curve \( \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} \) is called an integral curve.
The Damped Harmonic Oscillator

No external force $f(t) = 0$

The Phase Plane

The figure below shows the vector field and the integral curves for the initial value problem

$$\ddot{x} + c\dot{x} + x = 0 \quad x(0) = 1, \quad \dot{x}(0) = 0$$

for various damping constants.

$c = 3$ (over-damped)  $c = 2$ (critically-damped)

$c = 1$ (under-damped), and  $c = 0$ (undamped)