Lecture #2, Friday, Oct 2, 2015: 1st Order ODE's
Some definitions
How can we solve an ODE?
Linear ODE s:
Separable ODE's

Outline

1. Some definitions

2. Linear ODE s:

3. Separable ODE’s
   - Initial value problems:
Some definitions
How can we solve an ODE?
Linear ODEs:
Separable ODEs

Reading: §2.1, §2.2

Homework:
- §2.1 #13, 15, 18, 20, 31, 32.
- §2.2 #1, 4, 6, 29 (assume a, b, c, and d are all positive).
Some definitions

How can we solve an ODE?

Linear ODEs:

Separable ODEs

\[
\frac{dy}{dt} = f(t, y) \tag{1}
\]

A function \( y = \phi(t) \) is called a solution to (1) if the identity

\[
\phi'(t) = f(t, \phi(t))
\]

holds for all \( t \). The graph of a solution called an integral curve of the differential equation.

An initial value problem is a differential equation together with an initial condition

\[
y(t_0) = y_0
\]

for some numbers \( t_0 \) and \( y_0 \). The solution (to the initial value problem) is the function \( y = \phi(t) \) that satisfies the two conditions

\[
\phi'(t) = f(t, \phi(t)) \text{ and } \phi(t_0) = y_0.
\]
Example

The function \( \phi(t) = e^{-t^2/2} \) is a solution of the ODE \( y' = -ty \), because

\[
\phi'(t) = e^{-t^2/2} \left( -\frac{2t}{2} \right) = -te^{-t^2/2} = -t\phi(t).
\]

Another solution: \( \phi_1(t) = 5e^{-t^2/2} \).

In fact, the **general solution** is \( y = Ce^{-t^2/2} \).

To determine \( C \), need an **initial condition**.

**Example:** To find the solution of the **initial value problem**

\[
y' = -ty, \quad y(3) = 7,
\]

substitute into the general solution \( C e^{-3^2/2} = 7 \)

and solve for \( C \) to get \( y = 7e^{9/2} e^{-t^2/2} \).
In general, this is a difficult problem; but there are special cases where we can solve the ODE. We consider ODEs of the two forms:

- **first order (inhomogeneous) linear ODEs:**
  \[ y' + p(t)y = g(t), \]
  where \( p(t) \) and \( g(t) \) are continuous.
  When \( g(t) = 0 \) for all \( t \), the ODE is said to be **homogeneous**.

- **separable first order ODEs**
  \[ h(y)y' = g(t), \]
  where \( g \) and \( h \) are continuous functions.
Here are some examples of linear ODEs:

\[ \frac{dy}{dt} + 2y = 0 \quad y' + 2y = e^t \]
\[ \frac{dy}{dt} + ty = 0 \quad y' + ry = k, \quad r, k \text{ constant} \]
\[ (1 + t^2)y' + y = 0 \quad ty' + y = te^t \]
\[ y' + t^{-1}y = 1, \quad t > 0 \quad y' + y = \sin^{-1}(t), \quad |t| < 1 \]
\[ y' + \sqrt{1 - t^2}y = t, \quad |t| < 1 \]

Here are some examples of nonlinear ODEs:

\[ \frac{dy}{dt} + 2y^2 = 0 \quad y' + 2 \frac{1}{y} = e^t \quad y \frac{dy}{dt} + ty = 2 \]
\[ (y')^2 + y = t \quad y' + \sqrt{y} = 0 \quad y' + ty = (1 + y^2) \]
$y' + p(t)y = g(t).$

- Every solution is of the form: $y = y_p(t) + C y_1(t)$ where
  
  $C = \text{constant}$
  
  $y_p(t) = \underline{a \text{ particular solution}}$
  
  $y_1(t) = \text{solution of the homogenous equation: } y' + p(t)y = 0.$

- The solution of the \text{initial value problem}: 
  
  $y' + p(t)y = g(t) \quad y(t_0) = y_0$

  is $y = y_p(t) + C y_1(t)$ where $C$ is the solution of
  
  $y_p(t_0) + C y_1(t_0) = y_0.$
Set \( P(t) = \int_{t_0}^{t} p(s) \, ds \).

Notice that \( P'(t) = p(t) \), so \( \left( e^{P(t)} \right)' = e^{P(t)} P'(t) = e^{P(t)} p(t) \).

Multiply \( y' + p(t)y = 0 \) by the integrating factor \( e^{P(t)} \):

\[
e^{P(t)} y' + \left( e^{P(t)} \right)' y = 0
\]

By the product rule

\[
\left( e^{P(t)} y \right)' = e^{P(t)} y' + \left( e^{P(t)} \right)' y = 0
\]

So

\[
e^{P(t)} y = C \text{ or } y = Ce^{-P(t)}.
\]
Write solution in form:

\[ y = f(t)e^{-P(t)}, \text{ where } P(t) = \int^t p(s) \, ds \]

and \( f(t) \) is an unknown function of \( t \). Then

\[ \left( f(t)e^{-P(t)} \right)' + p(t)f(t)e^{-P(t)} = g(t). \]

Simplify:

\[ f'(t)e^{-P(t)} = g(t) \text{ or } f'(t) = g(t)e^{P(t)} \]

Therefore,

\[ f(t) = \int g(t)e^{P(t)} \, dt = \int^t g(s)e^{P(s)} \, ds + C \]

\[ y = f(t)y_1(t) = y_p(t) + Cy_1(t), \]

where \( y_1(t) = e^{-P(t)} \) and \( y_p(t) = e^{-P(t)} \int^t g(s)e^{P(s)} \, ds \).
Solve the ODE $y' + 3y = e^t$

- First solve the homogeneous ODE $y' + 3y = 0$ to get $y_1(t) = e^{-3t}$.
- Write $y = f(t)e^{-3t}$, and substitute into the inhomogeneous equation:

$$\left(f(t)e^{-3t}\right)' + 3f(t)e^{-3t} = e^t$$

Everything automatically collapses to give $A'(t)e^{-3t} = e^t$ or

$$A'(t) = e^{4t}$$

Integrate to get $f(t) = \frac{1}{4}e^{4t} + C$ and

$$y = f(t)e^{-3t} = \frac{1}{4}e^{4t}e^{-3t} + Ce^{-3t} = \frac{1}{4}e^t + Ce^{-3t}$$
Solve the initial value problem \( y' + 3y = e^t, \ y(1) = 6 \)

Find the “general solution” (as in the previous example)

\[
y = \frac{1}{4} e^t + C e^{-3t}
\]

Then plug in \( t = 1 \) and \( y = 6 \) to get

\[
6 = \frac{1}{4} e^1 + C e^{-3} \quad \text{or} \quad C = e^3 (6 - e/4)
\]

or \( y = \frac{1}{4} e^t + e^3 (6 - e/4) e^{-3t}. \)
Solve the ODE $y' + 3ty = te^{t^2}$

- First solve the homogeneous equation: $y' + 3ty = 0$ to get

  $$y_1(t) = e^{-3t^2/2}$$

- Next set $y = f(t)y_1(t) = f(t)e^{-3t^2/2}$ and plug into the inhomogeneous ODE to get

  $$A'(t)e^{-3t^2/2} = te^{t^2}$$ or $A'(t) = te^{5t^2/2}$

Integration gives $f(t) = \frac{1}{5}e^{5t^2/2} + C$ and

$$y = \left(\frac{1}{5}e^{5t^2/2} + C\right) e^{-3t^2/2} = \frac{1}{5}e^{t^2} + Ce^{-3t^2/2}$$
Because we actually **solve** for the solution of the initial value problem, it clearly exists and it is unique. The only assumption that we made is that the two integrals

\[
\int_{t_0}^{t} p(s) \, ds \quad \text{and} \quad \int_{t_0}^{t} e^{\int_{s}^{t} p(s) \, ds} g(s) \, ds
\]

make sense. This is the case if \( p(t) \) and \( g(t) \) are continuous functions. The precise statement of what we have proved is contained in the following theorem:

**Theorem**

Let \( p(t) \) and \( g(t) \) be continuous functions defined on the interval \( a < t < b \) and suppose that \( a < t_0 < b \). Then there is one and only one solution \( y = \phi(t) \) of the initial value problem

\[
y' + p(t) y = g(t), \quad y(t_0) = y_0.
\]

The solution is defined for all \( t \) in the interval \( (a, b) \).
Consider an ODE of the form

\[ h(y)y' = g(t) \]

where \( g(t) \) and \( h(y) \) are continuous functions. To solve let

\[
\int h(y) \, dy = H(y) + C_1 \quad \text{and} \quad \int g(t) \, dt = G(t) + C_2 .
\]

Then

\[
\frac{dG(t)}{dt} = g(t) \quad \text{and} \quad \frac{dH(y)}{dt} = h(y) \frac{dy}{dt} .
\]

So we get the **implicit** solution

\[
H(y) + C_1 = G(t) + C_2 . \quad \text{or} \quad H(y) = G(t) + C
\]

where \( C = C_2 - C_1 \), again an arbitrary constant.

**NOTE:** Solve for \( y \) in terms of \( t \) if you can!
Example:

Solve the differential equation

$$\frac{dy}{dt} = (1 + y^2)e^t.$$ 

Step 1. Get into the form $h(y)y' = g(t)$:

$$\frac{1}{1 + y^2} \frac{dy}{dt} = e^t.$$ 

Step 2. Integrate:

$$\int \frac{dy}{1 + y^2} = \int e^t \, dt.$$ 

$$\tan^{-1}(y) = e^t + C \text{ or } y = \tan(e^t + C)$$

where $C$ is an arbitrary constant.
To solve the initial value problem

\[ h(y)y' = g(t) \quad y(t_0) = y_0 \]

**Step 1.** Find the general solution:

\[ H(y) = G(t) + C \]

where \( H'(y) = h(y) \) and \( G'(t) = g(t) \).

**Step 2.** Find \( C \):

\[ H(y_0) = G(t_0) + C \text{ or } C = H(y_0) - G(t_0) \]

Final solution:

\[ H(y) - H(y_0) = G(t) - G(t_0). \]
Example: Solve $y' = (1 + y^2)e^t$, $y(0) = 1$

First solve the ODE:

$$\tan^{-1}(y) = e^t + C$$

Set $t = 0$ and $y = 1$ to get

$$\tan^{-1}(1) = e^0 + C \implies \pi/4 = 1 + C \implies C = \pi/4 - 1$$

So $\tan^{-1}(y) - \tan^{-1}(1) = e^t - e^0$

or $\tan^{-1}(y) - \pi/4 = e^t - 1$.

Solving for $y$ gives $y = \tan \left( e^t - 1 + \pi/4 \right)$. 