Lecture #6, Wed, Oct 14, 2015:

- **Reading**: §2.5
- **Homework**: §2.5 #7, 10, 22, 26 (Due Wed Oct 21)
The simplest model in which the reproduction rate decreases with increasing population is the logistic model where the reproduction rate decreases linearly with time:

\[
\frac{1}{P} \frac{dP}{dt} = R(P) = r\left(1 - \frac{P}{K}\right),
\]

where \(r\) and \(K\) are constants.

- For \(P\) is small, \(R(P) \approx r\) (intrinsic rate of growth)
  
  \(P(t) \approx P_0 e^{rt}\) for \(P\) small

- For \(P > K\), \(R(P)\) is negative and the population decreases with time. \(K\) called the carrying capacity of the environment.

- no matter what the initial value of \(P\), \(\lim_{t \to \infty} P(t) = K\).
Geometric Picture of the Logistic Equation
$P' = r(1 - P/K)P$ non-linear, but is separable:

$$\int \frac{dP}{(1 - P/K)P} = \int r dt$$

Integrate by partial fractions:

$$\frac{1}{(1 - P/K)P} = \left( \frac{1}{K - P} + \frac{1}{P} \right) \implies \int \frac{dP}{P(1 - P/K)} = \int \frac{dP}{K - P} + \int \frac{dP}{P}$$

So

$$\int \frac{dP}{P(1 - P/K)} = \int \frac{dP}{K - P} + \int \frac{dP}{P} = -\ln |K - P| + \ln |P| + C = \ln \left| \frac{P}{K - P} \right| + C$$

Thus $\ln \left| \frac{P}{K - P} \right| = rt + C \implies \left| \frac{P}{K - P} \right| = e^{rt+C} \implies \frac{P}{K - P} = Ae^{rt}. \text{ where } A = \pm e^C. \text{ So:}$

$$P = Ae^{rt}(K - P) = \frac{AKe^{rt}}{1 + Ae^{rt}} = \frac{K}{1 + e^{-kt}/A}$$

The graph of this function is an “S”-shaped curve called the Logistic Curve.
Notice that there are two constant solutions of the logistic equation

\[ \frac{dP}{dt} = r \left(1 - \frac{P}{K}\right) P \]

the functions \( P = 0 \) and \( P = K \). Also recall that the solution of the initial value problem

\[ \frac{dP}{dt} = r \left(1 - \frac{P}{K}\right) P, \quad P(0) = P_0 \]

approaches \( K \) as \( t \to \infty \) for all \( P_0 > 0 \). Thus, if the population is near \( K \), and suddenly deviates from that value (say due to a sudden and temporary shift in environmental variables) it will return to the value \( K \) at an exponential rate. For this reason, we say that \( P = K \) is a **stable equilibrium point**.
We can generalize as follows. Consider a differential equation of the form

\[
\frac{dy}{dx} = f(y)
\]

A value \( y = y_e \) is called an equilibrium point (it is also called a **critical point** or a **fixed point**) of the differential equation if \( f(y_e) = 0 \). Thus the fixed points are just the zeros of the function \( f(y) \).

**Example:** (a) The fixed points of the differential equation \( y' = (1 - y^2) \) are the points \( y = 1 \) and \( y = -1 \).
(b) The fixed points of the differential equation \( y' = \sin(\pi y) \) are the integers \( y = 0, \pm 1, \pm 2, \ldots \).
An equilibrium point that acts like an attractor is called a **stable** equilibrium point, otherwise it is called **unstable**. That is, “$y_e$ is a stable equilibrium point if the solution of the initial value problem approaches $y_e$ whenever the initial condition is sufficiently close to $y_e$.” An equilibrium point is called **unstable** if it is not stable.

You can find equilibrium points and tell which ones are stable directly from the direction field:

**Criterion for stability:** Suppose that $f(y)$ is negative for an interval above $y_e$ and positive for all $y$ in an interval below $y_e$. Then $y_e$ is a stable equilibrium point. Otherwise, $y_e$ is an unstable equilibrium point.
Consider the ODE \( y' = f(y) \) where the graph of \( f(y) \) is displayed below (flipped along the diagonal and aligned with the direction field):

**Question:** Where are the equilibrium points? Which ones are stable and which ones are unstable?