Math 307J Autumn 2015
Lecture 8: Introduction to 2nd order ODEs

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Wed, Oct 21, 2015
Lecture #8, Wed, Oct 21, 2015:

- Reading: §3.1
- Homework: §3.1 #1, 5, 6, 8, 9, 11, 12, 15, 16, 20. (Due Wed Oct 28)

1. Second Order ODE's
2. Linear Second Order Ordinary Differential Equations
3. Homogeneous ODE's
4. Solving the Initial Value Problem
These are ODE of the form

\[ \frac{d^2 y}{dt^2} = f \left( t, y, \frac{dy}{dt} \right). \]

Special cases that we can already solve:

- \[ \frac{d^2 y}{dt^2} = f \left( t, \frac{dy}{dt} \right). \]
- \[ \frac{d^2 y}{dt^2} = f \left( y, \frac{dy}{dt} \right). \]
Solving \( \frac{d^2y}{dt^2} = f \left( t, \frac{dy}{dt} \right) \)

- Let \( u = \frac{dy}{dt} \). Equation becomes
  \[
  \frac{du}{dt} = f(t, u),
  \]
  a first order differential equation, with solution, say \( u = U(t, C_1) \).

- Then
  \[
  \frac{dy}{dt} = U(t, C_1)
  \]
  and we can integrate to get
  \[
  y = \int_t^t U(s, C_1) \, ds + C_2
  \]

**Example:** \( y'' + t(y')^2 = 0 \). Let \( u = y' \). Then \( u' + tu^2 = 0 \). Can solve by separation of variables.
Solving \( \frac{d^2 y}{dt^2} = f \left( y, \frac{dy}{dt} \right) \)

- Let \( u = \frac{dy}{dt} \) and use the chain rule:

\[
\frac{d^2 y}{dt^2} = \frac{du}{dt} = \frac{du}{dy} \frac{dy}{dt} = u \frac{du}{dy}
\]

Differential equation then becomes the first order ODE

\[
u \frac{du}{dy} = f(y, u).
\]

- Let \( u = U(y, C_1) \) be the solution, then \( \frac{dy}{dt} = U(y, C_1) \).

- Solve by separation of variables:

\[
\int \frac{dy}{U(y, C_1)} = t + C_2.
\]

**Example:** Linear spring \( m\ddot{x} + kx = 0 \).
Linear second order ODEs

We are mainly interested in linear second order ordinary differential equations

\[ L[y] = \frac{d^2y}{dt^2} + p(t) \frac{dy}{dt} + q(t) y = g(t) \quad \text{(non-homogeneous)} \]

where we assume that the functions \( p(t) \), \( q(t) \) and \( g(t) \) are continuous on an interval \( a < t < b \).

Goal: To solve the equation for \( t \) in that interval.

Special case:

\[ L[y] = \frac{d^2y}{dt^2} + p(t) \frac{dy}{dt} + q(t) y = 0 \quad \text{(homogeneous)} \]

Remark: Similar to the linear first order ordinary differential equation:

\[ \frac{dy}{dt} + p(t) y = g(t). \]
Second Order ODE's
Linear Second Order Ordinary Differential Equations
Homogeneous ODE's
Solving the Initial Value Problem

Short-hand notation—Linear Operators

\[ L[y] = y'' + p(t)y' + q(t)y \]

**Example:** \( L[y] = y'' + ty' + e^t y \)

Then

\[ L[\sin(t)] = (\sin(t))'' + t(\sin(t))' + e^t(\sin(t)) \]
\[ = -\sin(t) + t\cos(t) + e^t\sin(t). \]

The homogeneous ODE: \( L[y] = 0 \)

The inhomogeneous equation: \( L[y] = g(t) \)

\( L \) called a **linear operator** because

\[ L[c_1 y_1(t) + c_2 y_2(t)] = c_1 L[y_1(t)] + c_2 L[y_2(t)] \]

for any two functions \( y_1(t) \) and \( y_2(t) \) and any constants \( c_1 \) and \( c_2 \).
Main Example: Damped Harmonic Oscillator

\[ m \ddot{x} + c \dot{x} + kx = f(t) \]

- **External force**: \( f(t) \)
- **Spring force**: \(-kx\)
- **Drag force**: \(-c \frac{dx}{dt}\)

**Parameters**:
- \( m \): mass
- \( k \): spring constant
- \( c \): damping constant

**Total force**:

\[ \text{total force} = -c \frac{dx}{dt} - kx + f(t) \]
**Initial value problems:**

The problem of finding a solution $y = \phi(t)$ of

$$L[y] = y'' + p(t) y' + q(t) y = g(t)$$

that satisfies the **TWO initial conditions**

$$y(t_0) = y_0 \text{ and } y'(t_0) = y'_0$$

for $a < t_0 < b$ is called an **initial value problem**.

**Theorem (Main Theoretical Result:)**

Suppose that $p(t)$, $q(t)$ and $g(t)$ are continuous on the interval $a < t < b$. Suppose further that $t_0$ is between $a$ and $b$. Then the initial value problem

$$L[y] = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0$$

has one and only one solution defined on the entire interval $a < t < b$.

**Note:** This does NOT tell us how to find the solution—it only confirms that a solution exists and that it is unique.
The Homogeneous case:

$L[y] = y'' + p(t)y' + q(t)y = 0, \ y(t_0) = y_0, \ y'(t_0) = y'_0$

Given two solutions, say $y_1(t)$ and $y_2(t)$, then the “linear combination”

$$y = c_1y_1(t) + c_2y_2(t)$$

is also a solution of $L[y] = 0$.
Therefore, once we have found two solutions $y_1(t)$ and $y_2(t)$, we can construct lots of solutions by taking linear combinations.
Solving:

\[ L[y] = 0 \quad y(t_0) = y_0 \quad y'(t_0) = y'_0. \]

- First find two specific solutions of the ODE
  \[ L[y] := y'' + py' + qy = 0. \]
  Call them \( y_1(t) \) and \( y_2(t) \).
- Form the **general solution** \( y = c_1 y_1(t) + c_2 y_2(t) \)
- Choose \( c_1 \) and \( c_2 \) so that the initial conditions are satisfied.

**To find** \( c_1 \) **and** \( c_2 \) **solve**

\[
\begin{cases}
  c_1 y_1(t_0) + c_2 y_2(t_0) = y_0 \\
  c_1 y_1'(t_0) + c_2 y_2'(t_0) = y'_0
\end{cases}
\]
Example

Solve the initial value problem

\[ L[y] = y'' - 3y' + 2y = 0 \text{ and } y(0) = 0, \quad y'(0) = 1. \]

- Look for solutions of the form \( y = e^{rt} \): If we substitute this into the equation \( L[y] = 0 \) we get
  \[ L[e^{rt}] = (r^2 - 3r + 2) \cdot e^{rt} = 0. \]

So \( r \) satisfies the quadratic equation

\[ r^2 - 3r + 2 = 0 \text{ or } (r - 2)(r - 1) = 0. \]

Therefore, \( r_1 = 1 \) and \( r_2 = 2 \) so \( y = C_1 e^t + C_2 e^{2t} \)

- Find \( C_1 \) and \( C_2 \):

  \[ \begin{align*}
  y(0) &= 0 \implies C_1 + C_2 = 0 \\
  y'(0) &= 1 \implies C_1 + 2C_2 = 1.
  \end{align*} \]

Thus, \( C_1 = -1 \) and \( C_2 = 1 \) and

\[ y = -e^t + e^{2t}. \]