Midterm #1 (50 points total)

Instructions: Please read the following instructions before you start the test:

• Show all work. Answers without sufficient work may not get full credit. Give all answers in exact form, unless otherwise stated—do not use a calculator.
• Be sure to either erase or scratch out any work that you do not want graded—two answers for a problem will nullify your solution, even if one of them is correct.

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TOTAL (50 points)
(1) **(10 points)** Find the solution of the initial value problem

\[ u' - 2xu = e^{x^2}, \quad u(1) = 0. \]

Simplify your answer.
(2) **(7 points)** Find the general solution of the differential equation $y' = x e^{x^2} (y^2 + 1)$. Solve for $y$ in terms of $x$, and simplify as much as you can.
(3) (8 points) Solve the initial value problem \( \frac{dy}{dt} = t\sqrt{\frac{4 - y^2}{4 + t^2}}, \ y(0) = 0 \). Solve for \( y \) in terms of \( t \), and simplify as much as you can.
(4) (10 points) After much experimentation, biologists have determined that the biomass $B(t)$ (measured in millions of pounds) of a certain species of plant $t$ years after 2000 is governed by a first order differential equation of the form

$$\frac{dB}{dt} = f(t, B).$$

In 2000 ($t = 0$), the biomass was measured to be 0.5 million pounds. The direction field of the differential equation is shown below (units along the horizontal axis are years, units along the vertical axis are millions of pounds).

(a) (7 points) On the figure below, carefully sketch the graph of $B(t)$.

(b) (3 points) What is your prediction for the total biomass in the year 2003? Give a brief explanation of how you arrived at your prediction.
(5) **(15 points)** At time $t = 0$ a container has 500 grams of salt dissolved in 100 liters of water. Pure water is pumped into the container at a constant rate of 10 liters/min. The well-mixed solution drains from the container at a rate $r(t) = 10 - 2t$ liters/min. for $0 \leq t \leq 5$ minutes.

**(a) (5 points)** Write down an initial value problem for the volume $V(t)$ of water in the tank at time $t$ for $0 \leq t \leq 5$ and solve it to obtain a formula for $V(t)$.

**(b) (5 points)** Use the result of (a) to write down an initial value problem satisfied by $y(t)$, the amount of salt in the container at time $t$, for $0 \leq t \leq 5$.

**(c) (5 points)** Solve the initial value problem that you found in part (b) to obtain a formula for $y(t)$. 
Answers

(1) \( (e^{-x^2}u)' = 1 \implies e^{-x^2}u = x + C. \) 
    \( u(1) = 0 \implies 1 + C = 0 \implies u(x) = (x - 1)e^{x^2} \)

(2) \( \int \frac{dy}{y+1} = \int e^{x^2} x \, dx \implies \tan^{-1} y = e^{x^2}/2 + C \implies y = \tan\left(e^{x^2}/2 + C\right) \)

(3) \( \int \frac{dy}{\sqrt{4-y^2}} = \int \frac{t \, dt}{\sqrt{4+t^2}} \implies \sin^{-1}(y/2) = \sqrt{4+t^2} + C. \) 
    \( y(0) = 0 \implies C = -2 \implies y = 2\sin\left(\sqrt{4+t^2} - 2\right) \)

(4) (a) ![Graph](image)

(b) About 2.75 million pounds. The graph drawn in part (a) is an approximation of the graph of \( B(t) \). The question asks for \( B(3) \), which is the vertical coordinate of the point \( P \) on the graph.

(5) (a) \( V' = 2t, \) \( V(0) = 100 \implies V(t) = t^2 + 100. \)

(b) \( y' = -\frac{y}{V(t)} \cdot (10 - 2t) \) and \( y(0) = 500 \implies y' = \frac{2t - 10}{t^2 + 100} \cdot y \) and \( y(0) = 500. \)

(c) \( y = C \exp\left(\int \frac{2t-10}{t^2+100} \, dt\right) \implies y = C(100 + t^2)e^{-\tan^{-1}(t/10)}. \) 
    \( y(0) = 500 \implies C = 5 \implies y = 5(100 + t^2)e^{-\tan^{-1}(t/10)}. \)