(1) Solve each of the following differential equations and initial value problems.

(a) $y'' - 4y = 0$, $y(0) = 1$, $y'(0) = 1$.

(b) $y'' - 4y = e^{2t}$, $y(0) = 1$, $y'(0) = 1$.

(c) $y'' - 4y' = e^t$.

(d) $y'' + y = 2\sin(t)$, $y(0) = 0$, $y'(0) = 0$.

(e) $ty'' + y' = 1$, $y(1) = 0$, $y'(1) = 2$.

(f) $y'' + 3y = t^2 + 1$.

(g) $y'' + y' + y = e^t \sin(t)$.

(h) $y'' - 4y' + 4y = e^{2t}$.

(i) $\ddot{y} + 4y = 3\cos(t) + 4\sin(t)$, $y(0) = 0$, $\dot{y}(0) = 0$.

(j) $(t - 1)y'' - ty' + y = 0$ $(t > 0)$. One solution is $y_1 = e^t$.

(k) $4t^2y'' + y = 0$. One solution is $\sqrt{t}$.

(l) $y'' + y = \tan(t)$.

(m) $y'' + 2y' + y = \frac{e^{-t}}{t}$, $t > 0$.

(n) $y'' - 3y' + 2y = \cos(e^{-t})$.

(o) $t^2y'' + ty' - y = t$, $t > 0$. The function $y = t$ is a solution of the homogeneous differential equation $t^2y'' + ty' - y = 0$.

(2) The following problems are based on the material in Chapter 6.

(a) Find the Laplace transform of the function $f(t) = t \sin(2t)$.

(b) Find the inverse Laplace transform of the function $F(s) = \frac{2s + 1}{s^2 + 4s + 5}$.

(c) Find the inverse Laplace transform of the function $F(s) = \frac{2s + 1}{s^2 + 4}$.

(d) Find the inverse Laplace transform of the function $F(s) = \frac{1}{s(s^2 + 4s + 5)}$.

(e) Solve the initial value problem $y'' - y = u_4(t) - u_5(t)$, $y(0) = 1$, $y'(0) = 0$.

(f) Compute $y(\tau)$ where $y(t)$ is the solution of the the initial value problem $y'' - y = u_4(t) - u_5(t) + u_6(t)$, $y(0) = 0$, $y'(0) = 0$.

(g) Compute $y(10\pi)$ where $y(t)$ is the solution of the the initial value problem $y'' + y = \delta(t - \pi) - \delta(t - 2\pi) + \delta(t - 3\pi)$, $y(0) = 0$, $y'(0) = 0$.

(h) Let $f(t) = t$ and $g(t) = \sin(t)$ compute $f * g(t)$.

(i) Let $y(t)$ be the solution of the initial value problem $y'' + 9y = e^{-t^2}$, $y(0) = 0$, $y'(0) = 0$.

Express $y(t)$ as the convolution of $e^{-t^2}$ with another function. Do not attempt to compute the convolution!
Let an annulus, i.e. the area between two concentric circles, be described in polar coordinates by $1 \leq r \leq 2$. If the inner boundary is held at temperature $T = 50^\circ C$ and the outer boundary at $T = 100^\circ C$ for a long time so that the annulus reaches thermal equilibrium, the temperature $T$ of a point in the annulus will depend only on the distance $r$ from the center and it can be shown that it satisfies the differential equation

$$\frac{d^2 T}{dr^2} + \frac{1}{r} \frac{dT}{dr} = 0$$

Find the temperature distribution $T = T(r)$.

Consider a “U”-shaped tube filled with liquid Mercury as shown in the figure below. The radius of the tube is 1 centimeter (so its diameter is 2 centimeters). There are 500 grams of Mercury in the tube. Liquid Mercury has a mass density of 13.5 grams per cubic centimeter.

The mercury in the tube will oscillate with a certain period $T$, measured in seconds. Your task is to compute $T$ by completing the following steps:

(a) Let $y(t)$ be the height above its equilibrium position of the liquid surface at the left vertical segment of the tube. (At equilibrium, both surfaces are at the same height above sea level and $y = 0$. When $y(t) < 0$ the right surface is higher than the left surface.) The only force acting on the mass of fluid in the tube is due to gravity. Compute the total force on the fluid (vertical component only) and use that formula to find a linear, homogeneous, constant coefficient, second order differential equation for $y(t)$.

(b) Compute $T$ by solving the differential equation you found in the previous part.

A cylindrical log 6 inches in diameter, 10 feet in length, and weighing 50 pounds is placed vertically in a lake so that it is free to bob up and down. A 25 pound weight of negligible volume is attached to the bottom of the log so that it remains vertical. A rope attached to the lower end of the log holds it in place with 2 feet of the log are out of the water. Water weights about 62.4 pounds per cubic foot. At time $t = 0$ the rope is cut and the log starts rising.

(a) Assume that there is no water resistance. Let $y = y(t)$ denote the length of the portion of the log above water level at time $t$ after the rope is cut. Write down an initial value problem for $y$. 

Hint: First draw a good picture and decide what quantity you are going to measure. In deriving the differential equation use Archimedes’ principle:

*An object that is completely or partially submerged in a fluid is acted on by an upward (buoyant) force equal to the weight of the displaced fluid.*

(b) After the rope is cut, the log bobs up and down in a sinusoidal manner. What is the period of the oscillation?
6. A 1 kilogram mass is suspended from the end of a spring with a spring constant of 1 N/m (Newtons per meter). The mass is free to move up and down, \( y \) is the amount (measured in meters) that the spring is stretched and there is no gravity. In addition, there is a damping mechanism which exerts a force of \(-cy\) Newtons, where \( c \) is a constant.

(a) What value of \( c \) will make the system critically damped?

(b) If at time \( t = 0 \) the spring is not stretched and the mass is moving at a rate of 0.5 m/sec (meters per second), what is the formula for \( y(t) \). (Use the value of \( c \) obtained in part a).

(c) What is the maximum amount by which the spring will be stretched?

7. Suppose that you are designing a new shock absorber for an automobile. The car has a mass of 1000 kg (kilograms) and the combined effect of the springs in the suspension system is that of a spring constant of 20000 N/m.

(a) Before a damping mechanism is installed in the car, when the car hits a bump it will bounce up and down. How may bounces will a rider experience in the minute right after the car hits a bump?

(b) Your job is to design a damping mechanism which eliminates oscillations when the car hits a bump. What is the minimum value of the effective damping constant that can be used?

(c) Suppose that at time \( t = 0 \) the car hits a bump. Immediately before that time the car was not moving up and down and the effect of the bump is to add a vertical component to the speed of the car of 1.0 meter/sec. How high will the car rise above its equilibrium position if you design the system with the damping constant you found in part (b)?

8. A spring-mass system has spring constant 3 N/m (i.e. 3 Newtons per meter). A mass of 2 kg is attached to the spring and the motion takes place in a viscous fluid which offers a resistance (measured in Newtons) numerically equal to twice the magnitude of the instantaneous velocity (measured in meters per second).

Let \( u \) denote the displacement of the mass from its equilibrium position, If the system is driven by an external force of \( 3 \cos(3t) - 2 \sin(3t) \) N, determine the formula for \( u(t) \) ignoring all “transients”.

9. Consider the initial value problem

\[
y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0
\]

where

\[
f(t) = \sum_{k=0}^{\infty} \left( u_{2k+1}(t) - 2u_{(2k+1)+1}(t) + u_{(2k+1)+1}(t) \right)
\]

(a) Carefully draw the graph of \( f(t) \) for \( 0 \leq t \leq 6\pi \).

(b) Find a formula for \( F(s) \), the Laplace transform of \( f(t) \).

(c) Find \( Y(s) \), the Laplace transform of the solution \( y(t) \). Express your answer in the form

\[
Y(s) = \frac{H(s)}{1 - e^{-sT}}
\]

for appropriate \( T \) and \( H(s) \).

(d) Express \( Y(s) \) in the form

\[
Y(s) = H(s) \sum_{k=0}^{\infty} e^{-kTs}.
\]

(e) Find \( h(t) \), the inverse Laplace transform of \( H(s) \) (from part (c)) and use this to find a formula for the solution \( y(t) \).

10. Consider the initial value problem

\[
y'' + 0.2y' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0.
\]

where \( f(t) \) is the same as in the previous problems. Repeat steps (c–e) of the previous problem.