(1) The direction field for a differential equation of the form \( y' = f(y) \) is sketched below. Assume that \( f \) has a derivative for all \( y \), so the existence and uniqueness theorem applies.

On the figure above, sketch as best you can the solution of the initial value problem
\[ y' = f(y), \quad y(2) = -2. \]

If \( y = \phi(x) \) is the solution, give your best estimates of \( \phi(0) \) and \( \phi(10) \).

(2) Solve each of the following first order differential equations and initial value problems.

(a) \( y' - 2xy = e^{x^2} \)
(b) \( y' = (1 - x)(2 - x), \quad y(0) = 0 \)
(c) \( y' = ay(b - y), \quad y(0) = y_0 \), where \( a > 0 \) and \( b > 0 \) are constants.
(d) \( y' + 2y = e^{-3x} \)
(e) \( y' = 1 - y^2, \quad y(0) = 0 \)
(f) \( y' = \cos(x)(y^2 + 1) \)
(g) \( (1 + x^2) \frac{dy}{dx} + 2xy = x(1 + x^2) \quad y(1) = 1 \)
(h) \( \frac{dy}{dt} = \sqrt{L - y^2}, \quad L > 0. \)

(3) Let \( r \) be a constant and let \( g(t) \) be a continuous function of \( t \). The purpose of this problem is to study the solution of the initial value problem
\[ y' + ry = g(t), \quad y(t_0) = y_0 \]

(a) Verify that \( y_1(t) = e^{-r(t-t_0)} \) is a solution of the ODE \( y' + ry = 0 \) and that \( y_1(t_0) = 1 \).
(b) Let \( A(t) = \int_{t_0}^{t} e^{-r(t-t_0)}(s)g(s)ds \). Verify that \( y_p(t) = A(t)e^{-r(t-t_0)} \) is a particular solution of the differential equation \( y' + ry = g(t) \) and that \( y_p(t_0) = 0 \).
(c) Finally, verify that $y = y_p(t) + y_0 e^{-r(t-t_0)}$ is a solution to the initial value problem, $y' + ry = g(t), \ y(t_0) = y_0$.

(4) If $y(t)$ is the temperature of an object placed at time $t = 0$ in an environment with constant temperature $b$, then Newton’s law of cooling says that $y(t)$ satisfies the differential equation $\frac{dy}{dt} = -k(y - b)$, where $k$ is a constant depending on the object. Suppose you make a pudding for a dinner party and put it in the refrigerator at 6 p.m. ($t = 0$). Your refrigerator maintains a constant temperature of 40°F. The pudding will gel when it cools to 45°. If the pudding’s temperature when you put it in the refrigerator is 190° and when your first guest arrives at 7 p.m. you measure the temperature as a 100°.

(a) Solve the differential equation to find an expression for $y(t)$. Your expression will involve the constant $k$ and a constant of integration $C$.

(b) Use the information about the two temperature readings to find the values of $k$ and $C$.

(c) When is the earliest you can serve dessert?

(5) Newton’s law of cooling applies when a colder object heats up in a warmer environment. Suppose water at 55°F is pumped into a swimming pool on a 90°F summer day. After 2 hours the temperature of the water is 60°. In how many hours (assuming the outside temperature remains 90°) will the water reach a comfortable swimming temperature of 70°?

(6) Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 200 gal of a dye solution with a concentration of 1 g/gal. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 2 gal/min, the well-stirred solution flowing out at the same rate.

(a) Write a differential equation for $y$, the amount of dye in the tank after $t$ minutes. Your differential equation should be a differential equation for exponential decay. Because the water flowing in is fresh, there is no constant term in the differential equation, and as a result, the amount of dye decreases exponentially.

(b) Find $y(t)$ as a function of $t$.

(c) What is the “half-life of $y(t)$?

(d) How much time elapses before the concentration of dye in the tank reaches 1% of its original value?

(7) The velocity $v(t)$ of a falling body meeting air resistance proportional to its velocity satisfies the differential equation $m \frac{dv}{dt} = -kv + mg$ where $g = 32\text{ft/sec}^2$ is the magnitude of gravitational acceleration, $k$ is a constant which depends on the object (called the coefficient of air resistance) and $m$ is the mass of the object. When a 120 lb ($= mg$) parachutist drops from a plane (with zero initial velocity), she first falls with a small coefficient of air resistance $k = 1.2\text{lb-sec/ft}$. Five seconds later her parachute opens and $k$ jumps to $12.0\text{lb-sec/ft}$.

**Note:** The convention we are using here is that $v$ is positive when the body is descending and negative when it is ascending.

(a) Write a differential equation for $v(t)$ for during the first five seconds.

(b) Solve the differential equation to find an expression for $v(t)$ during the first five seconds.

(c) Graph $v(t)$ for the first 5 seconds and show by a dotted line what $v(t)$ would be over the next ten seconds if the parachute didn’t open.
(d) Write the differential equation for \( v(t) \) after the parachute opens.

(e) Solve the differential equation for \( v(t) \) after the parachute opens. Use the value of \( v(5) \) from part (b) as the initial data.

(f) Graph (on the same sheet of graph paper) \( v(t) \) for the five seconds after the parachute opens.

(g) Again on the same sheet of graph paper draw two horizontal lines indicating the terminal velocity of the parachutist in the case where the parachute opens and in the case where is does not open.

(8) Do problems 1, 3, 5, 7 and 8 on page 62 of Boyce & DiPrima.

(9) Consider the logistic population model

\[
\frac{dP}{dt} = -\frac{P^2}{500} + 2P
\]

for a species of fish in a lake, were \( t \) is measured in years and \( P(t) \) is the number of fish in the lake at time \( t \). Suppose that it is decided that fishing will be allowed in the lake, but it is unclear how many fishing licenses should be issued. Suppose the average catch of a fisherman with a license is 5 fish per year.

(a) What is the largest number of licenses that can be issued if the fish are to have a chance to survive in the lake?

(b) Suppose the number of fishing licenses in part (a) are issued. What will happen to the fish population—that is, how does the behavior of the population depend on the initial population?

(c) The simple population model above can be thought of as a model of an ideal fish population that is not subject to many of the environmental problems of an actual lake. For the actual fish population, there will be occasional changes in the population that were not considered in the building of the model. If the water level were high because of a heavy rainstorm, a few extra fish might be able to swim down a usually dry stream bed to reach the lake; or the extra water might wash toxic waste into the lake, killing a few fish. Given the possibility of unexpected perturbation of the population, not included in the model, what do you think will happen to the actual fish population if we fix the fishing level at the one determined in part (b)?

(10) Suppose that an object of mass \( m \) moves under the influence of gravity along the parabola \( y = x^2/2 \) with the \( y \)-axis pointing up and the \( x \)-axis vertical. Suppose that at time \( t = 0 \) the object is located at the point \((1, 1/2)\) and that \( dx/dt = 0 \). Use conservation of energy to show that

\[
\frac{dx}{dt} = \sqrt{g \sqrt{\frac{1-x^2}{1+x^2}}}
\]

Do not attempt to solve! (Note: This problem is more difficult than the previous problems.)