Instructions: Do all of the following problems before Monday morning. The exam on Monday will partly be based on them (or on problems similar to them).

(1) Solve each of the following differential equations and initial value problems.

(a) \( ty'' + y' = 1, \ y(1) = 0, \ y'(1) = 2. \)
(b) \( y'' + 3y = t^2 + 1. \)
(c) \( y'' + y' + y = e^t \sin(t). \)
(d) \( y'' - y' = e^t \)
(e) \( y'' - 4y' + 4y = e^{2t} \)
(f) \( \ddot{y} + 4y = 3 \cos(t) + 4 \sin(t), \ y(0) = 0, \ y'(0) = 0. \)
(g) \( \ddot{y} + 4y = 12 \cos(4t) - 12 \sin(4t), \ y(0) = 0, \ y'(0) = 0. \)

(h) \( (t - 1)y'' - ty' + y = 0 \quad (t > 0). \) One solution is \( y_1 = e^t. \)
(i) \( 4t^2y'' + y = 0. \) One solution is \( \sqrt{t}. \)
(j) \( y'' + y = \tan(t). \)
(k) \( y'' + 2y' + y = \frac{e^{-t}}{t}, \ t > 0 \)
(l) \( y'' - 3y' + 2y = \cos(e^{-t}) \)

(m) \( t^2y'' + ty' - y = t, \ t > 0. \) The function \( y = t \) is a solution of the homogeneous differential equation \( t^2y'' + ty' - y = 0. \)

(2) Let an annulus, i.e. the area between two concentric circles, be described in polar coordinates by \( 1 \leq r \leq 2. \) If the inner boundary is held at temperature \( T = 50^\circ C \) and the outer boundary at \( T = 100^\circ C \) for a long time so that the annulus reaches thermal equilibrium, the temperature \( T \) of at a point in the annulus will depend only on the distance \( r \) from the center and it can be shown that it satisfies the differential equation

\[
\frac{d^2T}{dr^2} + \frac{1}{r} \frac{dT}{dr} = 0
\]

Find the temperature distribution \( T = T(r). \)
A weight of mass \( m = 5 \text{ kg} \) is suspended from a spring with unknown spring constant \( k \). The weight is free to move up and down. Ignoring friction, its position relative to its equilibrium position satisfies a differential equation of the form

\[
m \ddot{u} + ku = 0
\]

where \( t \) denotes time measured in seconds. To find the spring constant, the spring is set in motion and the graph of \( u(t) \) plotted. The result is shown in the following figure (horizontal axis is \( t \), vertical axis is \( u \)):

(a) What is the spring constant \( k \)? *Your answer can most easily be expressed in terms of \( \pi \).*

In a subsequent experiment, an external force of the form \( F(t) = F_0 \cos(\omega t) \) is applied to the mass so that the function \( u(t) \) now obeys the differential equation

\[
m \ddot{u} + ku = F_0 \cos(\omega t)
\]

The graph of the position \( u(t) \) is shown in the following graph:

(b) Estimate as best you can the frequency \( \omega \) of the applied force.

(c) Estimate as best you can the amplitude \( F_0 \) of the applied force.
(4) A simple pendulum of mass \( m \) and length \( L \) is hinged at a point \( P \) (see figure). If the wheel at the left of the figure rotates at a rate of \( \omega \) radians/second it forces the point \( P \) to move periodically back and forth. For small angle \( \theta \) (where \( \sin(\theta) \approx \theta \)) the angle \( \theta \) satisfies the differential equation
\[
\frac{d^2\theta}{dt^2} + g\theta = A \sin(\omega t).
\]
Assume, for simplicity that \( L = 1 \) meter, \( A = 1 \) meter/sec\(^2\), \( \omega = 1 \) rad/sec and \( g = 9.8 \) meter/sec\(^2\).

Find the solution which satisfies the initial conditions \( \theta(0) = \dot{\theta}(0) = 0 \).

Be sure to clearly show how you arrive at your answer— in particular, show the general solution to the ODE and how you got it.

(5) A 1 kilogram mass is suspended from the end of a spring with a spring constant of 1 N/m (Newtons per meter). The mass is free to move up and down, \( y \) is the amount (measured in meters) that the spring is stretched and there is no gravity. In addition, there is a damping mechanism which exerts a force of \(-cy\) Newtons, where \( c \) is a constant.

(a) What value of \( c \) will make the system critically damped?
(b) If at time \( t = 0 \) the spring is not stretched and the mass is moving at a rate of 0.5 m/sec (meters per second), what is the formula for \( y(t) \). (Use the value of \( c \) obtained in part a).
(c) What is the maximum amount by which the spring will be stretched?

(6) Suppose that you are designing a new shock absorber for an automobile. The car has a mass of 1000 kg (kilograms) and the combined effect of the springs in the suspension system is that of a spring constant of 20000 N/m.

(a) Before a damping mechanism is installed in the car, when the car hits a bump it will bounce up and down. How many bounces will a rider experience in the minute right after the car hits a bump?
(b) Your job is to design a damping mechanism which eliminates oscillations when the car hits a bump. What is the minimum value of the effective damping constant that can be used?
(c) Suppose that at time \( t = 0 \) the car hits a bump. Immediately before that time the car was not moving up and down and the effect of the bump is to add a vertical component to the speed of the car of 1.0 meter/sec. How high will the car rise
above its equilibrium position if you design the system with the damping constant you found in part (b)?

(7) A spring-mass system has spring constant 3 N/m (i.e. 3 Newtons per meter). A mass of 2 kg is attached to the spring and the motion takes place in a viscous fluid which offers a resistance (measured in Newtons) numerically equal to twice the magnitude of the instantaneous velocity (measured in meters per second).

Let $u$ denote the displacement of the mass from its equilibrium position. If the system is driven by an external force of $3 \cos(3t) - 2 \sin(3t)$ N, determine the formula for $u(t)$ ignoring all “transients”. Express your answer in the form $u(t) = A \cos(\omega t + \phi)$. 