Math 380 ***HOMEWORK #7*** ***due Tue, November 23***

Written assignment (4 problems).

W1. Let x be a real number such that $|x| < 1$. Compute $\lim_{n\to\infty} \prod_{i=1}^{n} \left(1 + x^{2^i}\right)$.

W2. Let $\{x_n\}$ be a sequence such that $\lim_{n\to\infty} (x_n - x_{n-2}) = 0$. Show that $\lim_{n\to\infty} \frac{x_n - x_{n-1}}{n} = 0$.

W3. Let $f(x)$ be a positive valued function over the reals such that $f'(x) > f(x)$ for all x. For what k must there exists N such that $f(x) > e^{kx}$ for $x > N$?

W4. Given $a > 1$, find $\lim_{x\to\infty} \left(\frac{1}{x} \cdot \frac{a^x - 1}{a - 1}\right)^{\frac{1}{x}}$.

Bonus Problem.

B1. Try to answer **P0** when a_0 is generic. What are the difficulties? (This is open-ended; do your best.)

Presentation assignment (6 problems).

P0. Define the sequence (a_n) by the recurrence $a_{n+1} = 3 - \frac{1}{a_n}$ for all non-negative integers n (starting from some given a_0). Assume that $a_0 = 2$. Does (a_n) converge? What if $a_0 = -1$?

P1. Define the sequence (a_n) by $a_0 = \alpha$ and

$$a_{n+1} = a_n - a_n^2$$

for $n \geq 1$. Determine for which values of α the sequence converges.

P2. Find the limit

$$\lim_{n\to\infty} \left(\prod_{k=1}^{n} \left(1 + \frac{k}{n}\right)\right)^{\frac{1}{n}}.$$

P3. For a fixed, positive integer k, the n^{th} derivative of $\frac{1}{x^k - 1}$ has the form $\frac{P_n(x)}{(x^k-1)^{n+1}}$, where $P_n(x)$ is a polynomial. Find $P_n(1)$.

P4. Let $x_0 = 1$, and $x_{n+1} = x_n + 10^{-10^{x_n}}$ for all $n \geq 1$. What can we say about $\lim_{n\to\infty} x_n$?

P5. Let $n = 111\ldots11$ with 2010 digits, all ones. What is the 1006th digit after the decimal point of $\sqrt{n}$?