Written assignment (4 problems).

W1. Prove that if $f : \mathbb{Z} \to \mathbb{Z}$ satisfies $f(f(n)) = f(f(n+2)+2) = n$ for all integers n and $f(0) = 1$, then $f(n) = 1 - n$.

W2. Let p_n be the probability that two numbers selected at random, independently and uniformly, from $\{1, 2, \ldots, n\}$ sum up to a square (e.g., $1 + 3 = 2^2$). Find $\lim_{n\to\infty} p_n\sqrt{n}$.

W3. Let f be a function defined on all reals, with the exception of 0 and 1, such that

$$f(x) + f(1 - \frac{1}{x}) = x + 1$$

for all $x \in \mathbb{R} \setminus \{0, 1\}$. Find f.

W4. Let C be the circle of radius 1 centered at the origin. Choose a point P at random on the circle, and a point Q at random inside the circle. What is the probability that the rectangle with diagonal PQ and sides parallel to the axes lies entirely in or on the circle?

Bonus Problems (2 problems).

B1. Given an irrational number $p \in (0, 1)$, is there a coin-flipping game that, with probability 1, ends in a finite number of steps, and wins with probability p?

B2. Let $f : \mathbb{N} \to \mathbb{N}$ a function such that $f(n+1) > f(f(n))$, for all n. Prove that $f(n) = n$.

Presentation assignment (5 problems).

P1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(xy) = xf(y) + f(x)$.

P2. Let f and g be monic quadratic polynomials such that $f(100)+f(10)+f(1) = g(100)+g(10) + g(1)$. Find an x such that $f(x) = g(x)$. (Recall that "monic" implies that the coefficient of the highest order term is 1.)

P3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(f(x)) = x$, for all x. Show that there must be at least one point $x_* \in \mathbb{R}$ such that $f(x_*) = x_*$.

P4. Suppose you are given a set of n *biased* coins, such that the probability that the mth coin will land on "heads" is $\frac{1}{2m+1}$. If you flip all n coins independently exactly once, what is the probability that you get an odd number of heads? (Hint: try to use the same method as in class.)

P5. Let a be the diameter of a small coin. Given a chessboard C made of squares of size $d \times d$ and with delimiting lines of negligible width, throw the coin "at random" on the chessboard. By "at random" we mean that the center of the circular coin can fall uniformly on any point on the chessboard (the margin of the coin may be outside the chessboard, if the center is close to the edge). What is the probability that the coin will fall entirely within one of the individual $d \times d$ squares?