March 12, 2008

Info on Final: You will be allowed to bring in a hand-written, one-sided sheet of paper format A4, containing useful formulas (*NOT problems*). Absolutely no books or notebooks allowed.

Reading Assignment: Re-read chapters 6.1, 8.1.1, 8.1.2, 8.2.1, 8.2.2, all of 9, 10, 11, 12, 13, 15, 17.

Practice Problems: Go over all the problems in the homework and midterm, make sure you understand how to do them. In addition, solve the following 8 problems, which will be discussed in class on Friday the 14th. DON'T WAIT UNTIL FRIDAY! We will go over them quickly, so you will need to have developped a good understanding of them, by then.

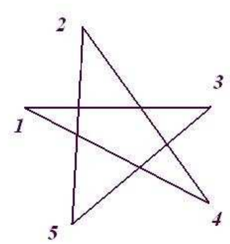
In addition, you may go over all solved exercises in the book.

Problem 1. A *connected* n-permutation is a permutation $p_1 p_2 \dots p_n$ for which there exists no $k < n$ such that $p_1 p_2 \dots p_k$ is a k-permutation. For example, 2314 is not connected, but 4321 is.

Let c_n be the number of connected n-permutations. Find a recurrence for c_n in terms of $c_1, \dots, c_{n-1}$.

Problem 2. Find an exponential generating function for the sequence given by the recurrence $a_n = na_{n-1} + n(n-1)a_{n-2}$, $a_0 = a_1 = 1$. Does this exponential generating function remind you of a famous sequence? (Hint: $1, 1, 2, 3, 5, \dots$) With this knowledge, solve for a_n.

Problem 3. How many automorphisms does the square have? How about the 5-star, below?



Problem 4. Prove that $E \leq 3V - 6$ for all simple connected planar graphs on more than 2 vertices. Find a planar graph for which $E > 3V - 6$.

Problem 5. (Problem 24 from Chapter 13) There are 9 passengers on a bus. Among any three of them, there are two who know each other. Show that there are 5 passengers who each know at least 4 of the people on the bus. (This is a little difficult.)

Problem 6. When we say that we randomly select two objects of the a certain kind, that means that we select an ordered pair (A, B) of of objects of the same kind. So (A, B) and (B, A) are different pairs, and $A = B$ is allowed.

- Randomly select two subsets of $[n]$. What is the probability that they have the same number of elements?
- Let $f(n)$ be te probability you were asked to compute above. Show that $\lim_{n\to\infty} f(n)\sqrt{\pi n} = 1$. (Hint: you will need the Stirling formula $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$.)

Problem 7. What is the expected number of k-cycles in a randomly selected $2k$-permutation? What about in a randomly selected $3k$-permutation? (Hint: this problem is a little tongue-in-cheek).

Problem 8. (Problem 11 from Chapter 17) Give a simple proof using graph theory for the fact that there is no algorithm that sorts n objects with less than $n - 1$ steps.

Problem 9. (Problem 18 from Chapter 17) The *distance* between two vertices x and y in a graph G is defined as the number of edges in the shortest path from x to y. The *diameter* of a graph G is defined as $\max\{d(x, y) \ : \ (x, y) \in G \times G\}$.

Find an algorithm that computes the diameter of a graph G on n vertices in $O(n^3)$ steps.

Problem 10. Given an irrational number $r > 0$, a famous number theory theorem says that there exists a sequence of positive integers $n_1, n_2, \ldots, n_k, \ldots$ such that $\lim_{k\to\infty} (n_k r - \lfloor n_k r \rfloor) = 0$.

Prove that $\sin n = O(1)$. Is $\sin n = \Theta(1)$? (Hint: π is an irrational number).