Math 462 ***HOMEWORK #3*** ***due Friday, January 25***

Reading Assignment: Read chapters 8.2.1, 8.2.2, 9.1, 9.2 (9.1 and 9.2 were studied last quarter; presumably you can just skim through and recall the important results).

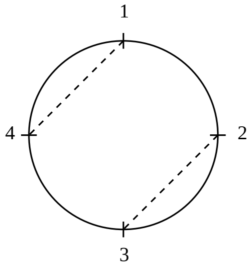
Written Assignment: There are 4 mandatory problems in this assignment, make sure you do them all. Problem 5 is optional (it earns "brownie points", which have no explicit value, but may put me in a particularly good mood when setting your grade).

Problem 1. Let P_n be the number of ways to place n open parentheses and n closed parentheses in a list in an "allowable" way (such that, among the first k parentheses, there are never more closed ones than open ones, for each $1 \leq k \leq 2n$). E.g., for $n = 4$, $((())())$ is allowable, but $())(()($ is not.

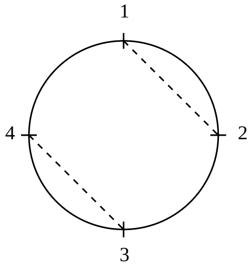
Show that $P_n = \frac{1}{n+1}\binom{2n}{n}$ (the Catalan numbers) ***by constructing a bijection*** between allowable lists of n closed and n open parantheses and Dyck paths of length $2n$.

Problem 2. Let there be $2n$ points on a circle. Pair up all points and draw the chords corresponding to each pair; we call the resulting picture an n-configuration.

Let T_n be the number of n-configurations in which no two chords intersect. For example, for $n = 2$, $T_n = 2$, as you can see below.

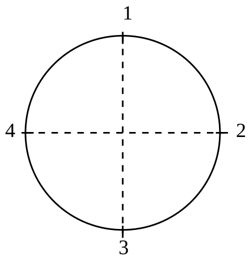


pairing (1,4) and (2,3)



pairing (1,2) and (3,4)

There is one more possible 2-configuration for the chords, but the two chords intersects in this case:



pairing (1,3) and (2,4)

Show that $T_n = \frac{1}{n+1}\binom{2n}{n}$ in ***one*** of two ways:

- either by showing that T_n satisfies the recurrence $T_{n+1} = \sum_{k=0}^{n} T_k T_{n-k}$, and that $T_0 = 1, T_1 = 1$.
- or by finding a bijection between configurations of n non-intersecting chords and either Dyck paths of length $2n$ or "allowable" ways of placing n closed and n open parentheses in a row (the latter are the objects of Problem 1).

Problem 3. This problem is based on Problem 19 from Chapter 6 (page 169).

(a) Given $r(n)$ as the number of n-permutations whose square is the identity, we have proved in Homework 2 that

$$r(n+2) = r(n+1) + (n+1)r(n) \ ,$$

with $r(0) = r(1) = 1$. We will now find the exponential generating function $R(x)$ for $r(n)$.

The solution to Problem 19, given on page 180, is unnecessarily complicated (do not look at it, lest you be lead astray). Multiplying the recurrence by $x^{n+1}/(n+1)!$ and summing over all $n \geq 0$, find a first-order differential equation satisfied by $R(x)$ and $R'(x)$. Solve it (like we did for the Bell numbers, in class).

b) Given a prime p, let $r(n)$ now be the number of n-permutations whose pth power is the identity. We have proved in Homework 2 that

$$r(n+p) = r(n+p-1) + (n+p-1)(n+p-2)\cdots(n+1)r(n) \ ,$$

with $r(0) = r(1) = \cdots = r(p-1)$. Again, let $R(x)$ be the exponential generating function for $r(n)$.

Adapt the calculation from part a) to find a first-order differential equation satisfied by $R(x)$ and $R'(x)$. Solve it (note that it should agree with what you found in part a), for p =2).

Problem 4. This problem is based on Problem 42 from Chapter 9 (page 201).

Let G be a simple graph on vertex set $[n]$ in which each vertex has degree two.

a) Prove that G is a union of disjoint cycles.

b) Let $g(n)$ be the number of graphs described above, and set $g(0) = 1, g(1) = 0, g(2) = 0$. Find a recurrence for $g(n)$.

Problem 5 (optional; "brownie points" problem). Find an infinite set of numbers n such that the complete graph K_n can be decomposed into edge-disjoint Hamiltonian cycles. For these numbers, indicate how to construct the decomposition.