Math 462 **Midterm Review Handout**

February 5, 2008

Info on Midterm: You will be allowed to bring in a hand-written, one-sided sheet of paper format A4, containing useful formulas (*NOT problems*). Absolutely no books or notebooks allowed.

Reading Assignment: Re-read chapters 61, 8.1.1, 8.1.2, 8.2.1, 8.2.2, all of 9, all of 10, 11.1-11.4.

Practice Problems: Go over all the problems in the homework, make sure you understand how to do them. In addition, solve the following 8 problems, which will be discussed in class on Wednesday. Don't wait until Wednesday! We will go over them quickly, so you will need to have developped a good understanding of them, by then.

In addition, you may go over all solved exercises in the book.

Problem 1. Let $a(n,k)$ be the number of permutations of length n with k cycles in which the entries 1 and 2 are *always* in the same cycle.

Prove that for $n \geq 2$,

$$\sum_{k=1}^{n} a(n,k)x^k = x(x+2)(x+3)\cdots(x+n-1) \ .$$

Problem 2. A group of tourists arrive at a restaurant. They split into groups, and then they sit down around circular tables, each group at a different table. (The tables are indistinguishable, and there are more than n of them.) Each group orders one of r possible drinks.

Prove that the number of ways this can happen is $r(r+1)\cdots(r+n-1)$. (Two seating arrangements are considered the same if each person has the same left neighbor in both of them.)

Problem 3. We color a set of $n \geq 1$ balls with four colors: red, blue, orange, and green, in such a way that the number of red balls is always even (may be 0) and the number of blue balls is always odd. Find the number of ways in which we can do this. (**Hint:** you will need to use the series for the sum and difference of e^x and e^{-x}.)

Problem 4. Let $C = \{c_1, c_2, \ldots, c_n\}$ be a set of positive integers. Let $g_C(n)$ be the number of n-permutations in which each cycle length belongs to C. Set $g_C(0) = 1$. Prove that

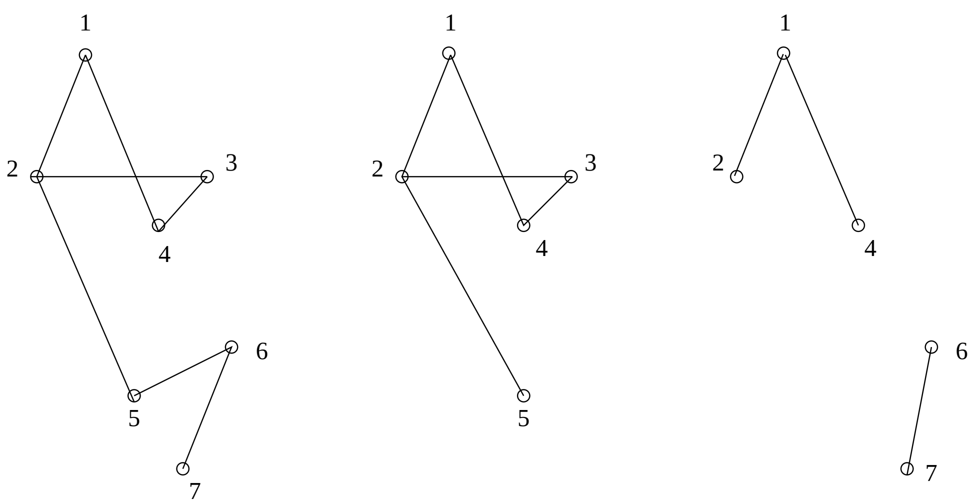
$$G_C(x) = \sum_{n\geq 0} g_C(n)\frac{x^n}{n!} = e^{\sum_{i=1}^{n} \frac{x^{c_i}}{c_i}} \ .$$

Hint: Look at the case when $n = 2$, and then generalize.

Problem 5. How many different labeled trees are there on $[n]$ that have no vertices with degree more than 2?

Problem 6. An *induced* subgraph of a graph G is a graph produced by the deletion of a set of vertices of G, together with their adjacent edges (see picture for an example).

A graph on 7 vertices (left) and the induced subgraphs obtained by deleting vertices 6 and 7 (middle), respectively 3 and 5 (right)



Prove that if a connected graph is not a complete graph, then every vertex of G belongs to some induced path on 3 vertices.

Problem 7. Let A be a matrix of 0s and 1s. Prove that there exist n entries equal to 1 such that each column and each row contains exactly 1 of them, if and only if any set k $(1 \leq k \leq n)$ of rows contain 1s in at least k columns. (See example below; the set of 1s is in red boldface.)

$$A = \begin{bmatrix} 0 & \mathbf{1} & 0 & 0 & 0 \\ 1 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 1 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 1 & 0 & 0 \end{bmatrix}$$

Problem 8. Prove that a tree always has more leaves than vertices of degree at least 3.