

February 8, 2008

NAME:

SIGNATURE:

**DO 4 OUT OF THE 5 PROBLEMS. IF YOU ATTEMPT ALL 5, MARK BELOW WHICH ONES YOU WANT GRADED, OTHERWISE WE WILL GRADE THE FIRST 4.**

Problem	Mark if you want it graded	Number of points	Points obtained
1		10	
2		10	
3		10	
4		10	
5		10	
Total		40	

**Instructions:**

- *Any attempt at cheating will be dealt with severely.*
- No books or notebooks allowed; you may use an  $8.5 \times 11$  double-sided, handwritten sheet of notes *for personal use* (do not share).
- Read problems *carefully*.
- Justify all your work.
- Raise your hand if you have a question.
- If you need more space, request additional blank sheets. It is your responsibility to have your instructor staple the additional sheets to your exam before you turn it in.
- Please turn off cell phones. GOOD LUCK!

**Problem 1.** (10pts) The parity of an  $n$ -permutation  $p$  is the parity of  $n - k$ , where  $k$  is the number of disjoint cycles in  $p$  (if  $n - k$  is even, the permutation is even, and if it's odd, the permutation is odd).

Let  $E_n$  be the number of even  $n$ -permutations, and  $O_n$  be the number of odd  $n$ -permutations. Prove that  $E_n = O_n = \frac{n!}{2}$ .

**Solution 1.** We use the unsigned Stirling numbers of the first kind formula:

$$\sum_{k=1}^n c(n, k)x^k = x(x+1) \cdot \dots \cdot (x+n-1). \quad (1)$$

Note that if  $n$  is even, then permutations with  $1, 3, 5, \dots, n-1$  cycles are all odd, whereas permutations with  $2, 4, 6, \dots, n$  cycles are all even. The reverse happens when  $n$  is odd.

Therefore showing that  $E_n = O_n = \frac{n!}{2}$  is tantamount to showing that

$$\sum_{k=0}^{\lfloor n/2 \rfloor} c(n, 2k+1) = \sum_{k=1}^{\lfloor n/2 \rfloor} c(n, 2k),$$

or, in other words, that

$$\sum_{k=1}^n c(n, k)(-1)^k = 0.$$

This follows immediately by making  $x = -1$  in formula (1).

**Solution 2.** We proceed by induction. First, let us prove the statement for  $n = 2$ . There are two permutations: the identity, which has parity the same as  $2 - 2 = 0$ , so it's even, and the transposition, which has parity  $2 - 1 = 1$ , so it's odd. Therefore the base case is true.

Let us now assume that the statement is true for  $n$ , and we shall prove it for  $n + 1$ . Assume that we have an even  $n + 1$ -permutation  $p$ ; then if  $k$  is the number of cycles in  $p$ ,  $n + 1 - k$  must be even.

Let's examine where  $p$  maps  $n + 1$ . Either  $n + 1$  is in a cycle by itself, which means that the  $n$ -permutation obtained from  $p$  by removing  $n + 1$  is, again, even ( $k$  becomes  $k - 1$ ,  $n + 1$  becomes  $n$ ,  $n - (k - 1) = n - k + 1$ ). On the other hand, what happens if  $n + 1$  is mapped into a cycle of length more than 1? It follows that removing  $n + 1$  from that cycle, we obtain an odd  $n$ -permutation (since it must have the same parity as  $n - k$ , which is odd).

So even  $n + 1$ -permutations are the result of adding one trivial cycle to even  $n$ -permutations, or inserting  $n + 1$  into a cycle in an odd  $n$ -permutation. We can do the former in only one way, but we can do the latter in  $n$  ways (before any of the  $n$  numbers).

Thus  $E_{n+1} = E_n + (n + 1)O_n$ , and by the induction hypothesis  $E_{n+1} = \frac{n!}{2} + (n + 1)\frac{n!}{2} = \frac{(n+1)!}{2}$ . This concludes the proof.

**Grading Key:** 10 for perfect solution; 7-9 for small mistakes or omissions; 4-6 for serious omissions; 1-3 for having the right idea but not going anywhere with it.

**Problem 2.** (10pts) Solve the recurrence  $a_n = 2na_{n-1} + n!$ ,  $a_0 = 1$ .

**Solution.** We use generating exponential functions; multiply the recurrence by  $x^n/n!$  and sum over  $n \geq 1$ ; if  $A(x)$  is the exponential generating function for the sequence  $\{a_n\}_{n \geq 0}$ , we get:

$$\begin{aligned}\sum_{n \geq 1} a_n \frac{x^n}{n!} &= \sum_{n \geq 1} 2x a_{n-1} \frac{x^{n-1}}{(n-1)!} + \sum_{n \geq 1} x^n, \\ A(x) - a_0 &= 2xA(x) + \frac{x}{1-x}, \\ A(x) - 1 &= 2xA(x) + \frac{x}{1-x}, \\ A(x)(1-2x) &= 1 + \frac{x}{1-x} = \frac{1}{1-x}, \\ A(x) &= \frac{1}{(1-x)(1-2x)}.\end{aligned}$$

We now write  $\frac{1}{(1-x)(1-2x)} = \frac{A}{1-x} + \frac{B}{1-2x}$ , and by solving the above we get  $A = -1$ ,  $B = 2$ . Therefore

$$A(x) = \frac{2}{1-2x} - \frac{1}{1-x} = 2 \sum_{n \geq 0} 2^n x^n - \sum_{n \geq 0} x^n = \sum_{n \geq 0} (2^{n+1} - 1)x^n.$$

Since  $A(x)$  is an exponential generating function, it follows that  $a_n = n!(2^{n+1} - 1)$ , which is easily verified by computing  $a_0 = 1$ ,  $a_1 = 3$ ,  $a_2 = 14$ , etc.

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**Problem 3.** (10pts) Prove that, in any *edge*-coloring of  $K_{17}$  with 3 colors, there exists a monochromatic triangle. If you want to use a result we proved in class, you must re-prove that result.

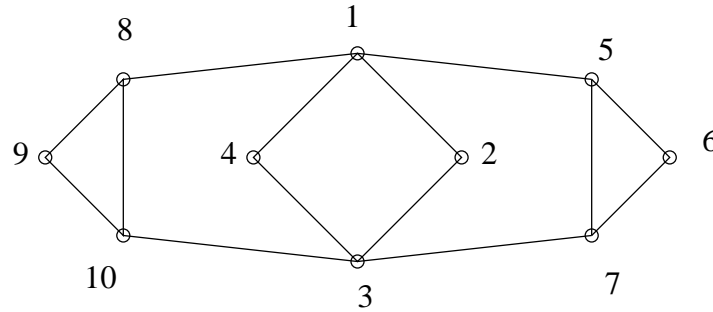
**Solution.** First we show that in any edge-coloring of  $K_6$  with 2 colors there is a monochromatic triangle.

This is done as follows: pick a vertex  $v$  in  $K_6$ ; it has degree 5, therefore by the Pigeonhole Principle three of its adjacent edges must have the same color, say R. The other three vertices adjacent to these edges (call them  $a, b, c$ ) are all connected; if they form a monochromatic B triangle, we're done; if not, at least one of the triangle edges (say  $ab$ ) must be R. But then  $v, a, b$  is a monochromatic R triangle, q.e.d.

Now back to our problem. Let  $v$  be a vertex in  $K_{17}$ ; of the 16 edges adjacent to  $v$ , by the Pigeonhole Principle, 6 must have the same color, say G. The other 6 vertices adjacent to these edges form a  $K_6$ . Either one of the edges in this  $K_6$  is also G, in which case we have found a monochromatic G triangle, or the  $K_6$  is edge-colored only with R and B. But then by the previous paragraph it must contain a monochromatic triangle.

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**Problem 4.** (10pts) How many automorphisms has the graph pictured below? Justify your answer (i.e. prove that there aren't any more/less; listing them is not enough).



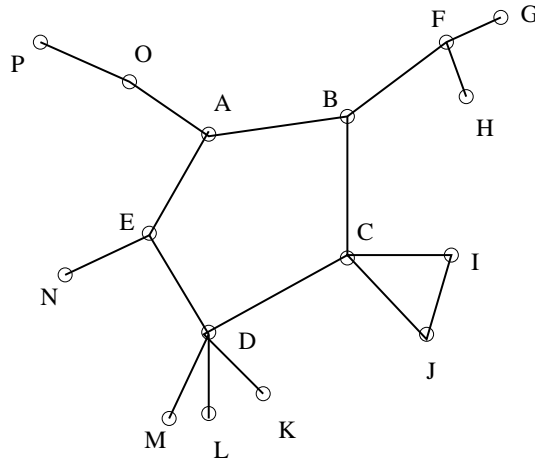
**Solution.** It is easy to see that the two vertices 6 and 9 can only be mapped to themselves or to each other (2 choices). Once we decide where to map 6, its adjacent vertices 5 and 7 can be mapped in 2 ways (if we keep 6 fixed, 5 and 7 could be swapped, and if we map 6 into 9, 5 could swap places with 8 while 7 swaps places with 10, or viceversa.) The choices we made on mapping 6, 5, 7 determine not only where we map 8, 9, 10, but also whether 1 and 3 stay put (if 5 was mapped into itself or into 8) or swap places (if 5 got mapped into 7 or 10).

Finally, 2 and 4 can stay put or swap places independently of what we've done so far (another 2 choices).

All in all, we have  $2 \cdot 2 \cdot 2 = 8$  mapping choices, so we have 8 automorphisms.

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**Problem 5.** (10pts) How many spanning trees does the graph pictured below have? Justify your answer (i.e. prove that there aren't any more/less).



**Solution.** There are two edge-disjoint cycles, ABCDE and CIJ. Note that removing exactly one edge from ABCDE and one edge from CIJ preserves the connectivity of the graph.

So by removing exactly one edge from ABCDE and one from CIJ we get a connected, cycle-free graph (i.e. a spanning tree).

On the other hand any spanning tree must be connected and cycle-free! So we *must* remove one edge in ABCDE and one in CIJ.

We have 5 ways of removing an edge from the first, and 3 ways of removing an edge from the second, and all choices are independent of each other. This yields a total of 15 spanning trees.

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