Nine problems for the UW Putnam Session. Issued 19 Oct. 2009

Play with a few of the problems below. If some are too easy (and you know how to solve them), move on – you're bound to hit some that will challenge you. These problems are mostly about Counting, Binomial Numbers, Graphs, Invariants, and the Principle of Inclusion-Exclusion.

If you don't know how to approach a problem, try a few small cases. Look for patterns. Draw a picture. Work Backward. Divide into cases. *Don't give up after 2 minutes.*

Binomial numbers. $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, with $m! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot m$ $(0! = 1)$. They count the number of ways in which you can choose k objects from among n. Useful identities:

a) $(x+y)^n = \sum_{i=1}^{n} \binom{n}{k} x^k y^{n-k}$ (binomial theorem)

b) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ (Pascal triangle identity)

Graphs. Graphs are collections of vertices and edges; a *planar* graph is a graph that can be drawn with no edge intersections (but drawing edges as arcs or curves rather than straight lines is allowable). For planar graphs, the "cells" obtained in the drawing are called *faces.*

Inclusion/Exclusion. Given the sets $A_1, \ldots, A_n$, we can compute the size of their union, $|\cup_{i=1}^n A_i|$, as follows:

$$|\cup_{i=1}^n A_i| = \sum_{i=1}^{n} |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| - \ldots - (-1)^n |A_1 \cap A_2 \ldots \cap A_n|$$

Here $|X|$ is the *c*ardinality (number of elements) of X, while $\cup$ and $\cap$ represent the union, respective intersection symbols.

Leftover problems from last time

Problem 11. Let $S_n = \{(p,q) \ : \ 0 < p < q \leq n, \ p+q > n, \ (p,q) = 1\}$. Show that $\sum_{(p,q)\in S_n} \frac{1}{pq} = \frac{1}{2}$.

Problem 13. Given $n+1$ integers between 1 and $2n$, show that one of them must be divisible by another. Is this best possible (can you show the same for n integers between 1 and $2n$)?

Problem 14. (Sperner's Lemma) Let $T = \Delta ABC$ be a triangle, and cut T into smaller triangles such that no vertex lying inside T sits on the edge of another triangle. (You might have vertices on the edges of T.) Color the vertices of all the triangles, subject to the following constraint: on edge AB you only use Red and Green; on edge BC you only use Green and Blue, and on edge CA you only use Blue and Red.

Show that there exists a triangle whose vertices are all of different colors.

New Problems

Problem 1 You are given a 4×4 board with four corners cut off. Is it possible for a knight to travel on all fields of the board exactly once and come back the field where he started?

Problem 2 Prove the following identities:

- $k\binom{n}{k} = n\binom{n-1}{k-1}$
- $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \ldots + \binom{k-1}{k-1}$
- $\binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}$. (Hint: the binomial theorem should come in handy.)

  This is a particular case of the Vandermonde formula; if you know it, try not to use it. Try to prove this problem in two ways, algebraically and combinatorially.

Problem 3. How many ways are there to place 8 identical rooks on the 8×8 chessboard so that no two attack each other? Same question for bishops (note that this time the question is not so easy).

Problem 4. Let V, E, F be, respectively, the number of vertices, edges, and faces of a planar graph. Prove that $V - E + F = 2$. Then use it to show that no complete graph with more than 4 vertices is planar.

Problem 5. A *fixed point* in a permutation π of $\{1, 2, \ldots, n\}$ is a value i such that $i = \pi(i)$. An *n-derangement* is a permutation of $\{1, 2, \ldots, n\}$ with no fixed points. How many n-derangements are there?

Problem 6. Prove that for $n = 6k + 5$,

$$\binom{m}{2} - \binom{m}{5} + \binom{m}{8} - \binom{m}{11} + \ldots - \binom{m}{m-6} + \binom{m}{m-3}$$

is never 0.