Fourteen Problems for the UW Putnam Session. Issued 5 Oct. 2009

Play with a few of the problems below. If some are too easy (and you know how to solve them), move on – you're bound to hit some that will challenge you. These problems are mostly about Induction, the Pigeonhole Principle, or Invariants.

If you don't know how to approach a problem, try a few small cases. Look for patterns. Draw a picture. Work Backward. Divide into cases. *Don't give up after 2 minutes.*

The Pigeonhole Principle. Given a coop with n holes and $kn+1$ pigeons outside, wanting to get in, some $k+1$ of them must use the same hole. Equivalently, given n boxes and $kn+1$ balls to be placed in the boxes, at least $k+1$ balls will go in the same box.

Induction. Let k be an integer, and $P(n)$ be a proposition (statement) about n for each integer $n \leq k$. Then if

1. $P(k)$ is true, and
2. for each integer $n \geq k$, if $P(n)$ is true, then $P(n+1)$ is true,

it follows that $P(n)$ is true for all $n \geq k$.

This principle allows us to prove an infinite number of propositions, in just two steps.

Invariants. Many "configuration" problems have some weight or quantity that does not change from configuration to configuration, and establishes whether or not a specific configuration is achievable through a set of moves from the initial configuration. It's a good idea to try and find this invariant.

Problem 1. Three integers are written on the board, and we play the following game: in each round we can replace two of the integers, say a and b, by (respectively) $\frac{3a-b}{2}$ and $\frac{3b-a}{2}$. (The order doesn't matter.) If you start out with $2007, 2008, 2014$, is it possible (after a finite number of rounds) to get $2009, 2010, 2011$?

Problem 2. Calculate $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2$ as a function of n (by induction).

Problem 3. Prove that $2!4!\ldots(2n)! \geq ((n+1)!)^n$, where $k! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot k$.

Problem 4. We label one of the vertices of the cube by 1, and all the others by 0. At each step, we pick an edge, and add one to the labels corresponding to its endpoints. Is it possible to get a configuration where

1. all labels are even?
2. all labels are divisible by 3?

Problem 5. Let $f(n)$ be the number of regions which are formed by n lines in the plane, where no two lines are parallel and no three meet at a point (e.g. $f(2) = 4$, $f(3) = 7$). Find a formula for $f(n)$.

Problem 6. Show that for $n \geq 6$, a square can be dissected into n smaller squares, not necessarily all of the same size.

Problem 7. Find the smallest number n such that no matter how we choose n points on the integer lattice (i.e., the set of points with integer coordinates in the plane), there are two whose midpoint is also on the lattice (has integer coordinates). What about if we look at the three-dimensional lattice? N-dimensional lattice?

Problem 8. You are given $n+2$ numbers. Prove that there are two numbers among them such that either their sum or difference is divisible by $2n$.

Problem 9. A grasshoper jumps along the line. His first jump is 1cm in length, the second is 2cm, the third is 3cm; the nth jump is n cm long. Can the grasshoper return to its initial position after 241 jumps? What about after 239 jumps?

Problem 10. You have chameleons of three different colors: White, Blue, and Red. Whenever two chameleons of different color meet, they both change their color to the third one. If you start out with 2010 white chameleons and 2009 blue chamelons, is it possible that after a while all chameleons will be red?

Problem 11. Let $S_n = \{(p, q) \;:\; 0 < p < q \leq n,\; p + q > n,\; (p, q) = 1\}$. Show that $\sum_{(p,q)\in S_n} \frac{1}{pq} = \frac{1}{2}$.

Problem 12. Let $A_1, A_2, \ldots, A_{1066}$ be subsets of a finite set of numbers X such that $|A_i| > \frac{1}{2}|X|$ for each i. Prove that there exist 10 numbers $x_1, x_2, \ldots, x_{10} \in X$ such that each A_i contains at least one of them. (Here $|S|$ is the size of the set S.)

Problem 13. Given $n + 1$ integers between 1 and $2n$, show that one of them must be divisible by another. Is this best possible (can you show the same for n integers between 1 and $2n$)?

Problem 14. (Sperner's Lemma) Let $T = \Delta ABC$ be a triangle, and cut T into smaller triangles such that no vertex lying inside T sits on the edge of another triangle. (You might have vertices on the edges of T.) Color the vertices of all the triangles, subject to the following constraint: on edge AB you only use Red and Green; on edge BC you only use Green and Blue, and on edge CA you only use Blue and Red.

Show that there exists a triangle whose vertices are all of different colors.