Math 462
HW 1 Solutions

Problem 2.21: Let $a_0 = 1$ and $a_{n+1} = 10a_n - 3$. Find an explicit formula for a_n.

Proof: We claim
$$a_n = \frac{2 \cdot 10^n + 1}{3}.$$
The proof is by induction on n. For the base case, when $n = 0$ we have
$$a_0 = \frac{2 \cdot 10^0 + 1}{3} = 1.$$
So now suppose the formula holds for a_n. Then
$$\begin{aligned} a_{n+1} = 10a_n - 3 &= 10\left(\frac{2 \cdot 10^n + 1}{3}\right) - 3 \\ &= \frac{2 \cdot 10^{n+1} + 10 - 9}{3} = \frac{2 \cdot 10^{n+1} + 1}{3}. \end{aligned}$$
Thus by induction the result holds for all n. □

Note: In general in this class it will never suffice to just state a formula as true. While we can use inspection to conjecture what the formula is, a proof is required to show that it holds for all values of n. Sometimes a formula will hold for the first million or so values and appear to be true, but fails for the million and first.

Problem 3.32: A student needs to work five days in January. He does not want to work on more than one Sunday. In how many ways can he select his five working days? (Assume January has 5 Sundays in the year in question.)

Proof: The student works either 0 Sundays or 1 Sunday, which are mutually exclusive events. There are $31 - 5 = 26$ non-Sundays in January, so there are $\binom{26}{5}$ ways for him to work 0 Sundays. If the student works 1 Sunday, there are $\binom{5}{1}$ ways to choose that Sunday, and $\binom{26}{4}$ ways to choose the remaining 4 days to work. Thus there are $\binom{5}{1}\binom{26}{4}$ ways for the student to work exactly 1 Sunday. This gives a total of $\binom{26}{5} + \binom{5}{1}\binom{26}{4}$ ways for the student to select his work schedule. □

Problem 4.30: The sum of each row of a 10×6 matrix in 36. If each column of the matrix has the same sum r, what is that sum?

Proof: Here we use the method of counting the same quantity, but in two different ways. The quantity to count is the sum of all the entries of the matrix. Summing by rows we get $10 \cdot 36 = 360$. Summing by columns gives $6 \cdot r$. These two quantities must agree, so $6r = 360$. Solving for r gives $r = 60$. □

Note: It does not suffice to merely construct the matrix with every entry equal to 6. While this matrix does satisfy the desired conditions, it is not the only such matrix. As an example consider the matrix

$$\begin{bmatrix} 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 \\ 8 & 5 & 5 & 8 & 5 & 5 \\ 5 & 8 & 5 & 5 & 8 & 5 \\ 5 & 5 & 8 & 5 & 5 & 8 \end{bmatrix}.$$

You must show that any matrix satisfying the above conditions will have columns summing to $r = 60$.