Math 462
HW 2 Solutions

Note: As a general comment, for a numerical solution that you derived using a combinatorial argument, unless specifically told to simplify, leave it in its original form. For example, if you have an answer that is a product of binomial coefficients, leave it as such. It is much easier to see where the answer came from when left in that form.

Note: Be very careful with using "..." in a proof. In most cases that is not rigorous enough for this class. Consider using something like induction to prove these kinds of statements. The same type of remark holds anytime you feel inclined to use phrases such as "we can see that the pattern is..."

Problem 1: Find formulas for the following:

(a) $c(n,1)$

Proof: If there is only one cycle, then it must be an n-cycle. So by Theorem 6.9 there are $\frac{n!}{n-1} = (n-1)!$ such permutations. $\square$

(b) $c(n, n-3)$

Proof: If there are $n-3$ cycles, then we can have either a 4-cycle, or a 3-cycle and a 2 cycle, or 3 2-cycles, and all the remaining cycles must be 1-cycles. Using Theorem 6.9 in each of these cases gives

$$c(n, n-3) = \frac{n!}{(n-4)!4} + \frac{n!}{(n-5)!2 \cdot 3} + \frac{n!}{(n-6)!3!2^3}.$$

$\square$

(c) $c(n,2)$

There are 2 ways to solve this problem. One method is a direct combinatorial method using Theorem 6.9 and the other uses the recurrence relation of Theorem 6.12.

Proof 1: If a permutation can only have two cycles and one of them is a k-cycle, the other must be an $(n-k)$-cycle. Here $1 \leq k \leq n-1$. We can count the number of permutations of this form using Theorem 6.9. If we sum over all such k, we will count every such permutation twice. Hence

$$c(n,2) = \frac{1}{2} \sum_{k=1}^{n-1} \frac{n!}{k(n-k)}.$$

$\square$

Proof 2: From Theorem 6.12 and part (a) above we have

$$c(n,2) = c(n-1,1) + (n-1)c(n-1,2) = (n-2)! + (n-1)c(n-1,2).$$

Dividing both sides by $(n-1)!$ and setting $a_n = \frac{c(n,2)}{(n-1)!}$ gives

$$\frac{c(n,2)}{(n-1)!} = \frac{1}{n-1} + \frac{c(n-1,2)}{(n-2)!}$$

$$a_n = \frac{1}{n-1} + a_{n-1}.$$

Since $a_1 = 0$, solving this recurrence relation gives

$$a_n = \sum_{k=2}^{n} \frac{1}{k-1} = \sum_{k=1}^{n-1} \frac{1}{k}.$$

Hence

$$c(n,2) = (n-1)!a_n = (n-1)! \sum_{k=1}^{n-1} \frac{1}{k}.$$

$\square$

Note: Theorem 6.9 is a very useful theorem. It tells you how many permutations have a particular cycle decomposition. Feel free to use this theorem instead of rederiving it. In general solutions to this problem were way too long because they did not make use of this theorem.

Problem 2: Let p be a prime number. Find a recursive formula for the number $r(n)$ of n-permutations whose p^{th} power is the identity.

Proof: Let π be an n-permutation. Then if π^p is the identity, pi can only consist of p-cycles and 1-cycles. (Think about how you would prove this.) So to determine $r(n+1)$ consider what type of cycle $n+1$ is in. If $n+1$ is in a 1-cycle by itself, then removing that cycle gives an n-permutation, which still has the property that its p^{th} power is the identity. If $n+1$ is in a p-cycle, then there are $\binom{n}{p-1}(p-1)!$ such p-cycles, and the remaining cycles must form a partition of $n+1-p$ whose p^{th} power is still the identity. Thus adding these two possibilities together gives

$$r(n+1) = r(n) + \binom{n}{p-1}(p-1)!r(n+1-p).$$

For the base cases we have $r(1) = 1$ and $r(m) = 0$ for $m \leq 0$. $\square$

Note: Any time you give a recurrence relation be sure to include enough base cases to properly define the relation for all values of n. A lot of people forgot to do this.

Problem 6.37(a): Prove that in the polynomial

$$(1+x)(1+2x)\cdots(1+(n-1)x)$$

the coefficient of x^{n-k} is $c(n,k)$ for all $k \in \{0,\ldots,n\}$.

Proof 1: From Lemma 6.13 we know

$$\sum_{k=0}^{n} c(n,k)x^k = x(x+1)\cdots(x+n-1)$$

is the generating function for the $c(n,k)$. So the trick will be to manipulate this polynomial so that it looks like the polynomial in the problem statement. To this end, replace $x \mapsto \frac{1}{x}$ to get

$$\sum_{k=0}^{n} c(n,k)x^{-k} = \tfrac{1}{x}(\tfrac{1}{x}+1)\cdots(\tfrac{1}{x}+n-1).$$

Then multiply both sides by x^n giving

$$\sum_{k=0}^{n} c(n,k)x^{n-k} = (1+x)\cdots(1+(n-1)x).$$

Now two polynomials are equal if and only if their corresponding coefficients are equal. So comparing coefficients gives the desired result. □

Proof 2: Show that the coefficients in the above polynomial satisfy the same recurrence as $c(n,k)$. Be sure to check the base case as well. □

Problem 7.14: How many n-permutations $p = p_1 p_2 \cdots p_n$ are there in which at least one of p_1 and p_n is even?

Proof: We will first count the complement of this set, namely the number of permutations where both p_1 and p_2 are odd. Note that the number of odd numbers in $\{1, \ldots, n\}$ is given by $\lceil \frac{n}{2} \rceil$. So there are $\lceil \frac{n}{2} \rceil$ choices for p_1, $\lceil \frac{n}{2} \rceil - 1$ choices for p_n, and the remaining $n-2$ elements $p_2, \ldots, p_{n-1}$ can be arranged in any way. Thus the total number of permutations where p_1 and p_n are both odd is given by

$$\left\lceil \frac{n}{2} \right\rceil \left(\left\lceil \frac{n}{2} \right\rceil - 1 \right) (n-2)!.$$

So the number of permutations where at least one of p_1 and p_n is even is given by

$$n! - \left\lceil \frac{n}{2} \right\rceil \left(\left\lceil \frac{n}{2} \right\rceil - 1 \right) (n-2)!.$$

□

Problem 8.23: Find an explicit formula for a_n if $a_0 = 1$ and $a_{n+1} = 3a_n + 2^n$ for $n \geq 0$.

Proof: Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for the sequence $\{a_n\}_{n=0}^{\infty}$. If we multiply both sides in the recurrence relation by x^{n+1} we get

$$a_{n+1}x^{n+1} = 3a_n x^{n+1} + 2^n x^{n+1} = 3x a_n x^n + x(2x)^n.$$

Summing over all n and solving for $F(x)$ then gives

$$\begin{aligned} \sum_{n=0}^{\infty} a_{n+1}x^{n+1} &= 3x \sum_{n=0}^{\infty} a_n x^n + x \sum_{n=0}^{\infty} (2x)^n \\ F(x) - a_0 &= 3xF(x) + \frac{x}{1-2x} \\ F(x)(1-3x) &= \frac{x}{1-2x} + 1 = \frac{1-x}{1-2x} \\ F(x) &= \frac{1-x}{(1-2x)(1-3x)}. \end{aligned}$$

Performing partial fractions then gives

$$F(x) = \frac{-1}{1-2x} + \frac{2}{1-3x} = \sum_{n=0}^{\infty} (2 \cdot 3^n - 2^n) x^n.$$

Hence a_n, which is the coefficient of x^n in $F(x)$, is $a_n = 2 \cdot 3^n - 2^n$. □

Note: If you derived your formula for a_n using generating functions there is no need to use induction to then prove your answer is correct. The proof that it is correct comes from the way you derived it. (Although it never hurts to get more practice using induction.)