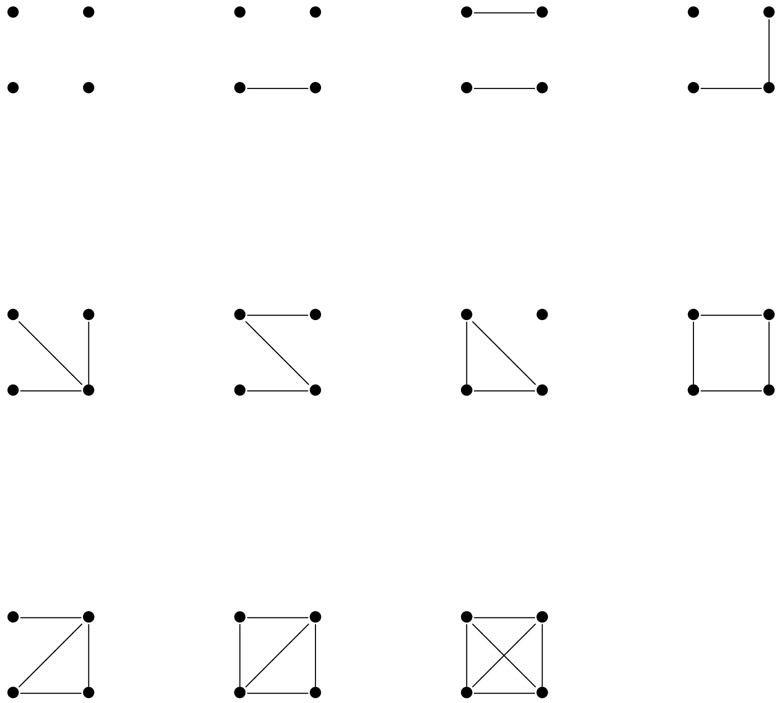
Math 462
HW 4 Solutions

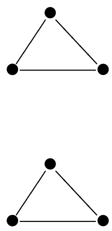
Problem 1: Find all non-isomorphic graphs on 4 vertices.

Proof: There are 11 such non-isomorphic graphs:

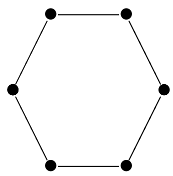


$\square$

Note: In general you must be careful when describing a graph in terms of a degree sequence. For graphs on $n = 4$ vertices the degree sequence of a graph is unique. However, for graphs on larger vertex sets this need not be true. As an example, consider the two graphs below, both of which have degree sequence $(2, 2, 2, 2, 2, 2)$:



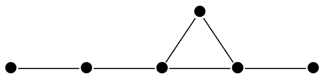
and



.

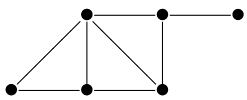
Problem 2: Find a simple graph on 6 vertices with no non-trivial automorphisms. As a bonus, show that no such graph exists with less than 6 vertices.

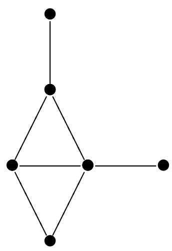
Proof: Here is one of the graphs that I came up with, although there are several others.

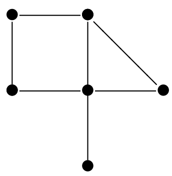


The main issue with this problem was that most people did not give an explanation for why their graph had no non-trivial automorphisms. For the graph pictured above, not that an automorphism would have to send a vertex of degree 1 to a vertex of degree 1, a vertex of degree 2 to a vertex of degree 2, and a vertex of degree 3 to a vertex of degree 3. But performing any of these exchanges would screw up the degrees of the adjacent vertices. Hence we can only have the trivial automorphism.

Here are a couple of the graphs that other people came up with:







For the bonus problem it is really just a matter of checking all possible isomorphism classes of graphs on 2, 3, 4, and 5 vertices. The case for $n = 4$ can be seen by looking at the solution for Problem 1 above. For $n = 5$ I think it is easiest to break up cases based on the largest vertex degree in the graph. $\square$

Problem 3:

(a) Let G be a simple graph on the vertex set $[n]$. Let $g(n)$ be the number of all such graphs. Set $g(0) = 1$, $g(1) = g(2) = 0$. Find the exponential generating function $G(x)$ for $g(n)$.

Proof: We have seen that $g(n)$ has the recurrence relation

$$g(n+3) = (n+2)g(n+2) + \binom{n+2}{2} g(n).$$

So similar to last week we multiply both sides by $\frac{x^{n+2}}{(n+1)!}$ and sum over all n to get

$$\begin{aligned}
\sum_{n=0}^{\infty} g(n+3)\frac{x^{n+2}}{(n+2)!} &= \sum_{n=0}^{\infty} g(n+2)\frac{x^{n+2}}{(n+1)!} + \sum_{n=0}^{\infty} \frac{g(n)}{2}\frac{x^{n+2}}{n!} \\
G'(x) - g(1) - xg(2) &= x\left(G'(x) - g(1)\right) + \frac{x^2}{2}G(x) \\
\frac{G'(x)}{G(x)} &= \frac{x^2}{2(1-x)} = -\frac{1}{2}x - \frac{1}{2} + \frac{1}{2(1-x)}.
\end{aligned}$$

Solving this differential equation and checking the initial conditions to determine the constant of integration gives

$$G(x) = \frac{1}{\sqrt{1-x}} e^{-\frac{x^2}{4} - \frac{x}{2}}.$$

□

(b) Explain why $G(x)$ is not $\frac{1}{1-x}$, which is the exponential generating function for the number of n-permutations, even though n-permutations are also unions of disjoint cycles on the set $[n]$.

Proof: One reason the two are not equal is that our graphs do not have cycles of length 1 or 2. Also, the cycles in the graphs can be read clockwise or counter-clockwise, both of which give the same graph. However, if you reverse the order of the elements in the cycle of a permutation, this gives a different permutation.

□

Note: Just like last week, when you have a generating function where the powers of x do not start at x^0, you must be very careful when you decide which terms you need to subtract from $G(x)$.

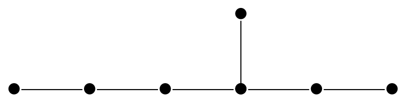
Problem 4: Prove that in any tree T, all longest paths intersect each other at the same vertex.

Proof: We first show that in any tree T two longest paths must intersect somewhere. To see this, suppose there are two paths $a = \{a_1, \ldots, a_n\}$ and $b = \{b_1, \ldots, b_n\}$ that don't intersect. Then since a tree is connected there exist two vertices a_r and b_s such that there is a path $p = \{a_r, v_2, \ldots, v_{k-1}, b_s\}$ from a_r to b_s that does not use any other vertices in a or b. By writing a or b in reverse order, we may assume $r, s \geq \frac{n}{2}$. But then the path $\{a_1, \ldots, a_r, v_2, \ldots, v_{k-1}, b_s, \ldots, b_n\}$ has length at least $\frac{n}{2} + k + \frac{n}{2} > n$, contradicting the fact that a and b were longest paths. Note that two paths can only intersect in at most one point or else a cycle would be formed.

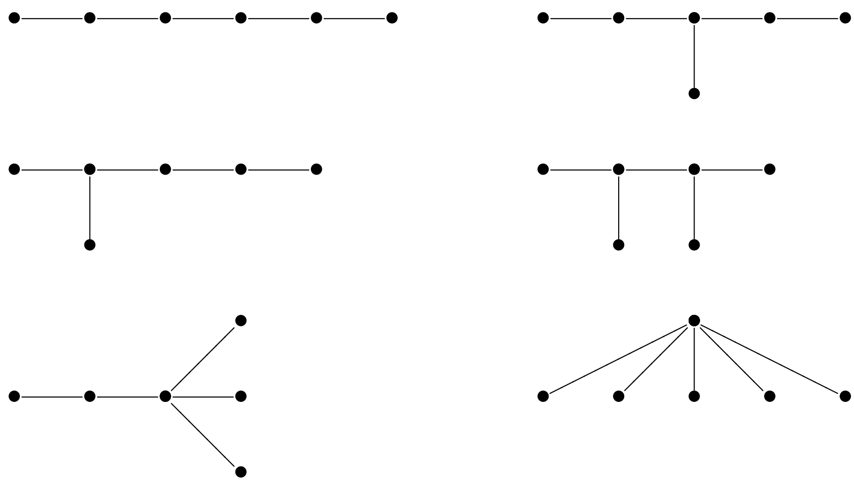
Now if there are only two longest paths, then they intersect each other at the same vertex. So suppose there are at least three longest paths in T. Call them a, b, and c. If they do not intersect each other at the same vertex, then there exist vertices x, y, and z such that a and b intersect at x, b and c intersect at y and a and c intersect at z. If any of the pairs of vertices are not distinct, then we have two paths intersecting twice, which yields a cycle. If all three vertices are distinct we can also form a cycle, again contradicting the fact that T is a tree. Thus all longest paths must intersect each other at the same vertex. $\square$

Problem 5: Find the smallest tree with no non-trivial automorphisms.

Proof: Consider the following tree:



This tree has only the trivial automorphism since any such automorphism must interchange leaves. However, the lengths of all the paths from a leaf to the root node are all different, so not such exchange is possible. We now claim that 7 is the minimum number of vertices for such a tree. By Problem 2 we know that no such graph exists on fewer than 6 vertices, and hence no such tree with exist. So we must only check all the isomorphism type for trees on 6 vertices. By inspection we see that they all have non-trivial automorphisms.



We can see that these are all the automorphism types by considering the maximal degree of a vertex in the tree. □