MATH 125 H MIDTERM 2, v.A Winter 2008

February 28, 2008

NAME: SIGNATURE:

STUDENT ID #: QUIZ SECTION:

Problem	Number of points	Points obtained
1	10	
2	10	
3	10	
4	10	
5	10	
Total	50	
Bonus	5	

Instructions:

- ***Any attempt at cheating will be punished.***
- No books or notebooks allowed; you may use an 8.5×11 double-sided, handwritten sheet of notes *for personal use* (do not share).
- Place a box around your final answer to each question.
- No *graphing* calculators allowed (scientific calculators OK).
- *Answers with little or no justification may receive no credit.*
- ***Answers obtained by guess-and-check work will receive little or no credit, even if correct.***
- Read problems *carefully.*
- Attempt Bonus Problem *only after* you have worked out Problems 1-5.
- Raise your hand if you have a question.
- If you need more space, use additional blank sheets which will be provided by your TA. It is your responsibility to have him/her staple the additional sheets to your exam before you turn it in.
- Please turn off cell phones. GOOD LUCK!

Problem 1. (10pts)

(a) (5pts) Evaluate the indefinite integral

$$\int \frac{x^4 + 3x^2 - 1}{x^2 + 4}\, dx$$

Show your work.

Using long division, find that $x^4 + 3x^2 - 1 = (x^2 + 4)(x^2 - 1) + 3$ (2pts)

Rewrite the integral as $\int \left(x^2 - 1 + \frac{3}{x^2+4}\right) dx$ (1pt)

$$= \frac{x^3}{3} - x + 3\int \frac{1}{x^2+4}\, dx$$

Use the formula $\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$ with $a = 2$ to obtain (1pt)

$\int \frac{x^4+3x^2-1}{x^2+4}\, dx = \frac{x^3}{3} - x + \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$ (1pt)

(b) (5pts) Evaluate the definite integral

$$\int_0^1 \frac{1}{(4 - x^2)^{3/2}}\, dx$$

Show your work.

Substitute $x = 2\sin\theta$, $dx = 2\cos\theta d\theta$, and note that (1pt)
$1 = 2\sin\theta$ implies $\theta = \pi/6$ to get

$$\int_0^{\pi/6} \frac{2\cos\theta\ d\theta}{(4-4\sin\theta)^{3/2}} = \int_0^{\pi/6} \frac{2\cos\theta}{(4\cos^2\theta)^{3/2}}\, d\theta \quad \text{(1pt)}$$

$$= \int_0^{\pi/6} \frac{2\cos\theta}{(2\cos\theta)^3}\, d\theta$$

$$= \int_0^{\pi/6} \frac{1}{4\cos^2\theta}\, d\theta \quad \text{(1pt)}$$

$$= \int_0^{\pi/6} \frac{1}{4}\sec^2\theta\ d\theta \quad \text{(1pt)}$$

$$= \frac{1}{4}\tan\theta\Big|_0^{\pi/6}$$

$$= \frac{1}{4}\frac{1}{\sqrt{3}} \quad \text{(1pt)}$$

Problem 2. (10pts)

(a) (5pts) Does the improper integral below converge? Explain.

$$\int_2^3 \frac{1}{x^2-2x}\,dx$$

First, note that the function has an infinite discontinuity at $x=2$, and is continuous on $(2,3]$, so we're looking at a Type II improper integral. (1pt)

Next, write $\frac{1}{x^2-2x} = \frac{A}{x-2} + \frac{B}{x}$, and solve to get $A=2$, $B=-2$. (1pt)

Integrate $\int_t^3 \left(\frac{2}{x-2} - \frac{2}{x}\right)\,dx = (2\ln|x-2| - 2\ln|x|)\Big|_t^3$ (1pt)

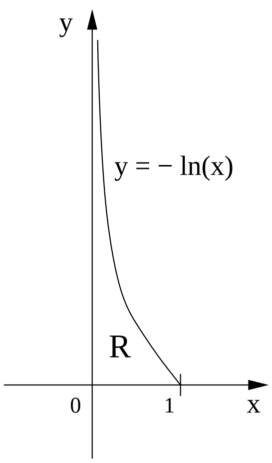
$= 2\ln 1 - 2\ln 3 - 2\ln(t-2) + 2\ln t$

$= 2\ln t - 2\ln(t-2) - 2\ln 3.$ (1pt)

Now note that $\lim_{t\to 2^+} 2\ln t - 2\ln(t-2) - 2\ln 3 = 2\ln 2 - 2\ln 3 - \lim_{t\to 2^+} 2\ln(t-2) = \infty.$

Hence $\int_2^3 \frac{1}{x^2-2x}\,dx = \lim_{t\to 2^+}\int_t^3 \frac{1}{x^2-2x}\,dx = \infty,$ (1pt)
and the integral is divergent.

(b) (5pts) Consider the unbounded region R pictured below, lying between the curve $y=-\ln x$, the x-axis, and the y-axis. Is the area of region R finite or infinite? If it's finite, compute it.



The area of the region is given by $\int_0^1 -\ln x\,dx$, (1pt)
and since $-\ln x$ is continuous on $(0,1]$ but not at 0,
we're looking at a Type II improper integral.
We have to examine $\int_t^1 -\ln x\,dx$ and then let $t\to 0^+$.

Through integration by parts, $\int_t^1 -\ln x\,dx = -x\ln x\Big|_t^1 + \int_t^1 x\frac{1}{x}\,dx$ (1pt)

$= -x\ln x\Big|_t^1 + x\Big|_t^1$

$= t\ln t + 1 - t$ (1pt)

Now, $\lim_{t\to 0^+} t\ln t + 1 - t = 1 + \lim_{t\to 0^+} t\ln t$

$= 1 + \lim_{t\to 0^+} \frac{\ln t}{\frac{1}{t}}$

and by l'Hospital's Rule, $\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+}\frac{\ln t}{\frac{1}{t}} = \lim_{t\to 0^+}\frac{\frac{1}{t}}{-\frac{1}{t^2}} = \lim_{t\to 0^+} -t = 0.$ (1pt)

Therefore $\int_0^1 -\ln x\,dx = \lim_{t\to 0^+}\int_t^1 -\ln x\,dx = 1 + \lim_{t\to 0^+} t\ln t = 1$
and the integral converges to 1. (1pt)

Problem 3. (10pts) Compute the average of the function $f(x) = x\tan^{-1}(x)$ on $[0, 2]$.

The average of the function is $\frac{1}{2}\int_0^2 x\tan^{-1} x\, dx$ (2pts)
and through integrationg by parts, we get

$$I := \frac{1}{2}\int_0^2 x\tan^{-1} x\, dx = \frac{1}{2}\frac{x^2}{2}\tan^{-1} x\Big|_0^2 - \frac{1}{2}\int_0^2 \frac{x^2}{2}\frac{1}{x^2+1}\, dx \qquad \text{(2pts)}$$

Use the trig substitution $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$ to obtain

$$I = \tan^{-1} 2 - \frac{1}{4}\int \frac{\tan^2\theta}{\tan^2\theta+1}\sec^2\theta\, d\theta \qquad \text{(1pt)}$$

$$= \tan^{-1} 2 - \frac{1}{4}\int \tan^2\theta\, d\theta \qquad \text{(since } \tan^2\theta + 1 = \sec^2\theta\text{)} \qquad \text{(1pt)}$$

$$= \tan^{-1} 2 - \frac{1}{4}\int (\sec^2\theta - 1)\, d\theta \qquad \text{(since } \tan^2\theta = \sec^2\theta - 1\text{)} \qquad \text{(1pt)}$$

$$= \tan^{-1} 2 - \frac{1}{4}\tan\theta\Big|_{x=0}^{x=2} + \frac{1}{4}\theta\Big|_{x=0}^{x=2} \qquad \text{(1pt)}$$

$$= \tan^{-1} 2 - \frac{1}{4}x\Big|_0^2 + \frac{1}{4}\tan^{-1} x\Big|_{x=0}^{x=2} \qquad \text{(since } \theta = \tan^{-1} x\text{)} \qquad \text{(1pt)}$$

$$= \frac{5}{4}\tan^{-1} 2 - \frac{1}{2}. \qquad \text{(1pt)}$$

Problem 4. (10pts) (*This problem continues on the next page.*)

(a) (4pts) Set up an integral for evaluating the length of the curve given by $y = x^2 + 2x + 7$ between $x = 2$ and $x = 6$. DO NOT EVALUATE IT.

$$L = \int_2^6 \sqrt{1 + (2x+2)^2}\, dx \quad .$$

Two points for the correct formula, one point for correct bounds, one point for $\frac{dy}{dx} = 2x + 2$.

Problem 4. (cont'd)

(b) (6pts) Using Simpson's Rule with $n = 4$, estimate the integral you have found in part (a).

First we divide the interval $[2, 6]$ into 4 parts; this yields
$x_0 = 2,\ x_1 = 3,\ x_2 = 4,\ x_3 = 5,\ x_4 = 6.$ (1pt)
We get $\Delta x = \frac{6-2}{4} = 1.$ (1pt)

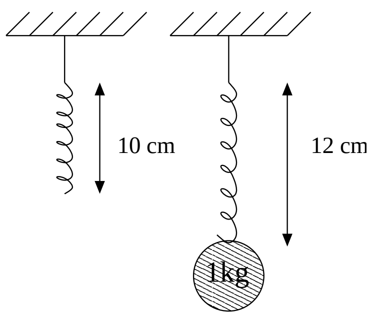
Simpson's Rule gives the approximation

$$S := \frac{\Delta x}{3}\left(f(2) + 4f(3) + 2f(4) + 4f(5) + f(6)\right) \quad \text{(2pts)}$$

$$= \frac{1}{3}\left(\sqrt{1+6^2} + 4\sqrt{1+8^2} + 2\sqrt{1+10^2} + 4\sqrt{1+12^2} + \sqrt{1+14^2}\right) \quad \text{(1pt)}$$

$$= \frac{1}{3}(\sqrt{37} + 4\sqrt{65} + 2\sqrt{101} + 4\sqrt{145} + \sqrt{197}) \approx 40.21 \quad \text{(1pt)}$$

Problem 5. (10pts) In the figure below you see a spring attached to the ceiling of a lab. The natural length of the spring is 10cm. The spring is stretched to 12cm by attaching to it a ball of mass 1kg. If a student pulls down on the ball, what is the additional work she needs to do to stretch the spring to 13cm?



First we find the force that pulled the string down 2cm:
$F = mg = 1 \cdot 9.8 = 9.8N.$ (3pts)
Since $F = k \cdot x$, where x is the displacement,
$9.8 = k \cdot 0.02$ wherefrom $k = 490.$ (3pts)

From here, all we need do is compute the work, keeping in mind that stretching the spring from 12cm to 13cm means stretching it from .02m to .03m away from its natural length: (1pt)

$$W = \int_{0.02}^{0.03} kx\,dx = \left.\frac{kx^2}{2}\right|_{0.02}^{0.03} \quad \text{(1pt)}$$

$$= \frac{490 \cdot 0.03^2}{2} - \frac{490 \cdot 0.03^2}{2} \quad \text{(1pt)}$$

$$= 0.1225 \text{ Joules} \quad \text{(1pt)}$$

Bonus Problem 6. (5pts; no partial credit.) Compute

$$\int \frac{1}{x^9 - x}\, dx.$$

(**Note:** the method of partial fractions is hopeless here. Try something else.)

Solution 1. Since $x^9 - x = x(x^8 - 1)$, try $u = x^8$, $u^{1/8} = x$, i.e. $\frac{1}{8}u^{-7/8}\, du = dx$. The integral becomes

$$\begin{aligned}
\int \frac{1}{x^9 - x}\, dx &= \int \frac{1}{u^{1/8}(u-1)} \frac{1}{8} u^{-7/8}\, du \\
&= \int \frac{\frac{1}{8}}{u^{1/8}(u-1)u^{7/8}}\, du \\
&= \int \frac{\frac{1}{8}}{u(u-1)}\, du \\
&= \frac{1}{8} \int \frac{1}{u(u-1)}\, du \\
&= \frac{1}{8} \int \left(\frac{1}{u-1} - \frac{1}{u-1} \right) du \\
&= \frac{1}{8} (\ln|u-1| - \ln|u|) + C \\
&= \frac{1}{8} \ln \frac{|x^8 - 1|}{x^8} + C\,.
\end{aligned}$$

Solution 2. We make the substitution $x^4 = \sec\theta$ to get $x = (\sec\theta)^{1/4}$, and $dx = \frac{1}{4}(\sec\theta)^{-3/4} \sec\theta \tan\theta\, d\theta = \frac{1}{4}(\sec\theta)^{1/4} \tan\theta\, d\theta$.
Thus,

$$\begin{aligned}
\int \frac{1}{x^9 - x}\, dx &= \int \frac{1}{4} \frac{(\sec\theta)^{1/4} \tan\theta}{(\sec\theta)^{1/4}(\sec^2\theta - 1)}\, d\theta \\
&= \frac{1}{4} \int \frac{\tan\theta}{\sec^2\theta - 1}\, d\theta \\
&= \frac{1}{4} \int \frac{\tan\theta}{\tan^2\theta}\, d\theta \\
&= \frac{1}{4} \int \frac{1}{\tan\theta}\, d\theta \\
&= \frac{1}{4} \int \cot\theta\, d\theta \\
&= \frac{1}{4} \ln|\csc\theta| + C \\
&= \frac{1}{4} \ln\left|\frac{\sqrt{x^8 - 1}}{x^4}\right| + C \\
&= \frac{1}{8} \ln\left|\frac{x^8 - 1}{x^8}\right| + C\,.
\end{aligned}$$