Solutions and Grading Key for the Final, M124, Au '07

Problem 1. (12pts)

(a) (3pts)

$$\frac{d}{dx}\left(\frac{\sec x}{1+x^4}\right) = \frac{(1+x^4)\frac{d}{dx}(\sec x) - (\sec x)\frac{d}{dx}(1+x^4)}{(1+x^4)^2} \qquad \text{correct quotient rule: 1pt}$$

$$= \frac{(1+x^4)(\sec x \tan x) - (\sec x)(4x^3)}{(1+x^4)^2} \qquad \text{2pts for the 2 derivatives; 1 each}$$

(b) (3pts)

$$\frac{d}{dx}(2^x \tan(3x)) = 2^x \frac{d}{dx}(\tan 3x) + \frac{d}{dx}(2^x)\cdot \tan(3x) \qquad \text{correct product rule: 1pt}$$

$$= 2^x \sec^2(3x)\cdot 3 + (2^x \ln 2)\cdot \tan(3x) \qquad \text{2pts for the 2 derivatives; 1 each}$$

(c) (3pts)

$$y = \ln(1+e^x)^x, \qquad y = x\ln(1+e^x) \qquad \text{correct setup: 1pt}$$

$$\frac{y'}{y} = \ln(1+e^x) + x\frac{e^x}{1+e^x} \qquad \text{correct product rule and differentiation: 1pt}$$

$$y' = (1+e^x)^x\left(\ln(1+e^x) + \frac{xe^x}{1+e^x}\right) \qquad \text{correct answer: 1pt}$$

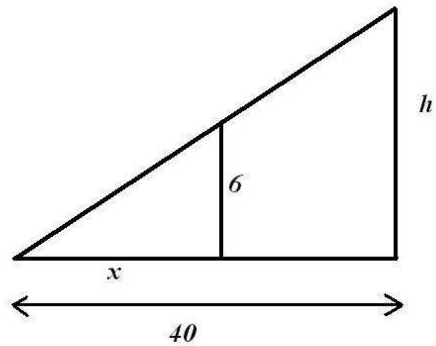
(d) (3pts)

$$\frac{d}{dx}\left(\sin^{-1}(\sqrt{4-x^2})\right) = \frac{1}{\sqrt{1-(\sqrt{4-x^2})^2}}\cdot\frac{1}{2\sqrt{4-x^2}}\cdot(-2x)$$

One point per each correct derivative (term) above, for a total of 3.

Problem 2. (9pts)

Here's the schematic picture of the situation; x is the distance form the lightsource, h is the height of the shadow.



Have: $\frac{dx}{dt} = 5$.

Want: $\frac{dh}{dt}$ when $x = 30$.

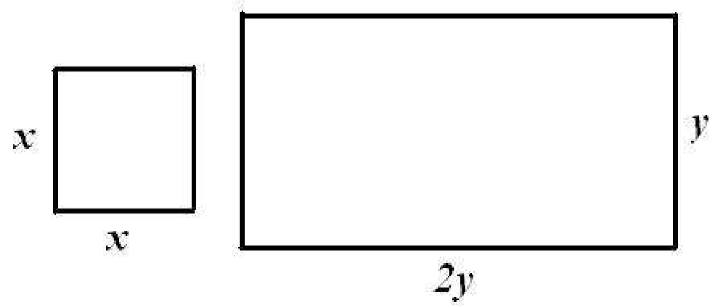
Equation (based on similar triangles): $\frac{h}{40} = \frac{6}{x}$, $h = 240x^{-1}$. (5pts)

Differentiate: $\frac{dh}{dt} = -240x^{-2}\frac{dx}{dt}$; plug in $x = 30$ to get the answer $-\frac{240\cdot5}{30^2} = -\frac{4}{3}$ft/s. (4pts)

No penalty for unsigned answer. 2 points off if 10, rather than 30, was plugged in. This was a related rates problem; if no differentiation occurred but answer was obtained as $\pm 4/3$, the student was awarded only 3-4 points out of 9.

Problem 3. (12 pts)

Here's a picture for the situation: two enclosures, one square and one rectangular with the large edge twice as big as the small.



(a) (7pts)

$$F = 136 = 4x + 6y \qquad (1pt)$$
$$A = x^2 + 2y^2 \qquad (1pt)$$
$$y = \frac{136 - 4x}{6} \qquad (1pt)$$
$$A = x^2 + 2\left(\frac{136 - 4x}{6}\right)^2 \qquad (1pt)$$
$$A' = 2x - \frac{16}{36}(136 - 4x) = 0, \qquad \text{therefore } x = 16 \qquad (1pt)$$

Any check that $x = 16$ is a minimum, for example using the closed interval method (note that $0 \le x$ and $0 \le y$ implies $0 \le x \le \frac{136}{4} = 34$) (1pt)

Enclosures should be 16×16 and 24×24 (1pt)

(b) As above, $0 \le x \le 34$. Since $A = x^2 + 2\left(\frac{136-4x}{6}\right)^2$,

$$x = 0, \qquad A(0) = 2\left(\frac{136}{6}\right)^2 \approx 1027.55 \qquad (2pts)$$
$$x = 34, \qquad A(34) = 34^2 = 1156 \qquad (2pts)$$

therefore, by the closed interval method, and since at the (only) critical point $x = 16$ we have a minimum, the maximum is achieved at $x = 34$.

Enclosures should be 34×34 and 0×0. (1pt)

Problem 4. (10pts)

(a) Total of 4 points.

Differentiate implicitly:

$4x + x \cdot \frac{dy}{dx} + y + 2y \cdot \frac{dy}{dx} = 0$ (2pts)

wherefrom

$\frac{dy}{dx} \cdot (x + 2y) = -4x - y$ (1pt)

thus

$\frac{dy}{dx} = \frac{-4x-y}{x+2y}$. (1pt)

(b) Total of 3 points.

Substitute 0 for x and 2 for y in the above
to get the slope of the tangent line:

$m = \frac{-4 \cdot 0 - 2}{0 + 2 \cdot 2} = -\frac{1}{2}$ (1pt)

Therefore the equation of the tangent is

$y - 2 = -\frac{1}{2} \cdot x$ or $y = -\frac{1}{2}x + 2$ (2pts)

(c) Total of 3 points.

Substitute $\frac{4\sqrt{7}}{7}$ for x and $-\frac{2\sqrt{7}}{7}$ for y in the above
to get the slope of the tangent line:

$m = \frac{-4 \cdot \left(\frac{4\sqrt{7}}{7}\right) + \frac{2\sqrt{7}}{7}}{\frac{4\sqrt{7}}{7} + 2 \cdot \left(-\frac{2\sqrt{7}}{7}\right)} = \frac{-\frac{14\sqrt{7}}{7}}{0}$ (1pt)

therefore the slope is undefined.
Since by inspection there are no “kinks” in the curve,
it follows that the tangent line at this point must be vertical. (1pt)

Thus the equation of the tangent line must be $x = \frac{4\sqrt{7}}{7}$. (1pt)

Problem 5. (12pts)

No partial credit.

(a)		$x = -2$	(2pts)
(b)		$x = 4$	(2pts)
(c)		$x = -4$	(1pt)
	and	$x = 1$	(1pt)
(d)		$[-6, -2]$	(2pts)
	and	$[4, 7]$	(2pts)
(e)		$[-4, 1]$	(2pts)

Problem 6. (12pts)

(a) (3pts, no partial credit) Answer: 10 (period is recognizable as $\frac{1}{10}$s, therefore 10 motions are completed in one second.

(b) (3pts, no partial credit) Answer: $t = \frac{1}{40}$s (since sin is maximized when its argument is $\pi/2$).

(c) (3pts) $v(t) = \frac{dx}{dt} = 60\pi \cos(20\pi t)$; therefore, when $t = \frac{1}{40}$, $v(t) = 0$. We awarded full credit for correctly computing the derivative at whatever value found in (b).

(d) (3pts) $a(t) = -1200\pi^2 \sin(20\pi t)$, which, when $t = \frac{1}{40}$, is equal to $-1200\pi^2 \text{cm/s}^2$. Again, we awarded full credit for correctly computing the derivative at whatever value found in (b).

Problem 7. (12pts)

(a) θ is the inverse tangent of the y-coordinate divided by the x-coordinate of the stone, i.e.,

$$\theta = \tan^{-1} \frac{180 - 16t^2}{4t} = \tan^{-1} \left(\frac{45}{t} - 4t \right) \qquad \text{(5pts)}$$

(b) Differentiate:

$$\frac{d\theta}{dt} = \frac{1}{1 + ((45/t) - 4t)^2} \left(-\frac{45}{t^2} - 4 \right) = -\frac{9}{10} \text{ rad/sec}$$

when $t = 3$.

Scoring: 7 points, total. If the wrong function was used in (a) (but one of comparable complexity) and if the derivative is correct, the student earned 7 points; however, if no chain rule was used, no credit was awarded for this part.

2 points were awarded for simplifying the answer (if evaluation occurred).

Problem 8. (12pts)

(a) Since the denominator $x^2 + 1 > 0$ for all x, (1pt)

there are no vertical asymptotes. (1pt)

(b)

$$\lim_{x\to\infty} \frac{4x}{x^2+1} = \lim_{x\to\infty} \frac{\frac{4}{x}}{1+\frac{1}{x^2}} = \frac{0}{1+0} = 0 \qquad \text{(1pt)}$$

therefore $y = 0$ is a horizontal asymptote. (1pt)

(c)

$$f'(x) = \frac{4(x^2+1) - 4x \cdot 2x}{(x^2+1)^2} = \frac{4(1-x^2)}{(x^2+1)^2} \qquad \text{(1pt)}$$

$$f'(x) = 0 \implies x = \pm 1 \qquad \text{(1pt)}$$

Consider the sign of $f'(x)$:

x		-1		1	
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	↘		↗		↘

(1pt)

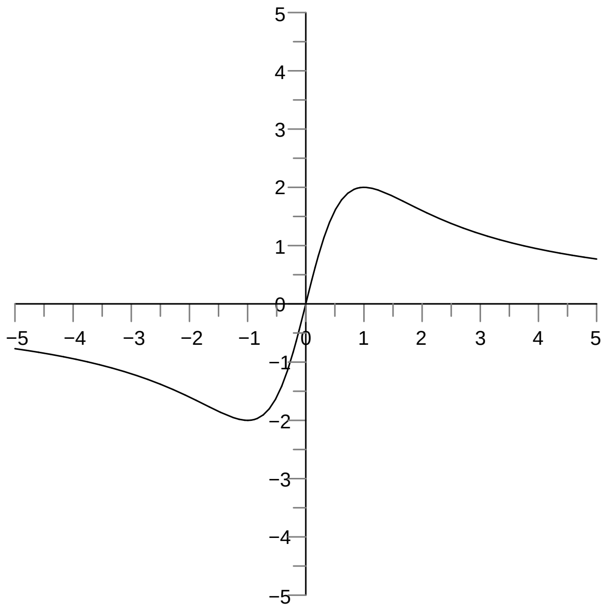
Therefore $x = -1$ is a local minimum, and $x = 1$ is a local maximum. (1pt)

(d)

$$\begin{aligned} f''(x) &= 4\left(\frac{-2x \cdot (x^2+1)^2 - (1-x^2) \cdot 2(x^2+1) \cdot (2x)}{(x^2+1)^4}\right) \\ &= \frac{8x}{(x^2+1)^3}\left(-(x^2+1) - (1-x^2) \cdot 2\right) \\ &= \frac{8x(x^2-3)}{(x^2+1)^3} \qquad \text{(1pt)} \end{aligned}$$

therefore the inflexion points are at $x = 0, \pm\sqrt{3}$. (1pt)

(e) Markers: $(0,0)$, $(-1,-2)$, $(1,2)$, $(-\sqrt{3},-\sqrt{3})$, $(\sqrt{3},\sqrt{3})$.



(2pts)

Problem 9. (9pts)

The linear approximation at $a = 0$ is given by

$$\begin{aligned} L(x) &= f(0) + f'(0)(x-0) && \text{(2pts)} \\ f(0) &= \tan\left(\frac{\pi}{4}e^0\right) = \tan\left(\frac{\pi}{4}\right) = 1 && \text{(1pt)} \\ f'(x) &= \sec^2\left(\frac{\pi}{4}e^x\right) \cdot \frac{\pi}{4} \cdot e^x && \text{(2pts)} \\ f'(0) &= \sec^2\left(\frac{\pi}{4}\right)\frac{\pi}{4} = \frac{\pi}{2} \approx 1.5708 && \text{(1pt)} \end{aligned}$$

Exact form or decimal expansion in the above or below were both fine.

$$\begin{aligned} L(x) &= 1 + \frac{\pi}{2}\,x \\ &= 1 + 1.5708\,x && \text{(1pt)} \end{aligned}$$

The linear approximation gives the following estimate for $f(0.2)$:

$$L(0.2) = 1 + \frac{\pi}{2} \times 0.2 = 1.3142\ . \qquad \text{(2pts)}$$