Putnam diagnostic test, October 12, 2009

Do as much as you can. No calculators or any other materials are allowed; only pen, pencil, and paper.

Problem 1. Given a triangle ABC and M the midpoint of BC, cut ABC into two pieces along AM (see Figure 1). Consider the triangles AMC and AMB. What is the smallest number of pieces that we can cut AMC into so that we can reassemble them into a triangle congruent to triangle AMB?

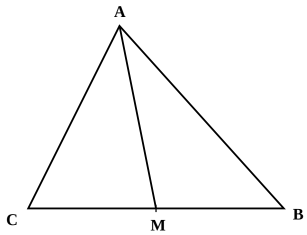

Figure 1: Triangle ABC, M midpoint of BC

Problem 2. The sequence a_n is defined by $a_1 = A$, $a_2 = B$,

$$a_{n+2} = \frac{a_n a_{n+1}}{2a_n - a_{n+1}}\ , \quad \text{for all } n \geq 2.$$

Assume that A and B are integers and that they are chosen so that the denominator in the expression is never 0. For what values of A and B does the sequence have the property that an infinite number of a_ns are integers?

Problem 3. How many real roots does the equation $2^x = 1 + x^2$ have?

Problem 4. Compare: $\prod_{i=1}^{25}(1 - \frac{i}{365})$ and $\frac{1}{2}$.

Problem 5. Given the parallelogram $ABCD$, let N be a point on CD, M be a point on BC, such that the areas of the triangles ABM, MNC, and ADN are as in Figure 2. Find the area of the parallelogram $ABCD$.

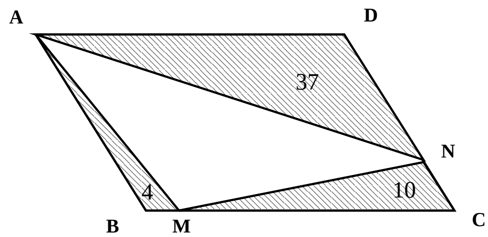

Figure 2: Parallelogram $ABCD$, $Area(ABM) = 4$, $Area(MNC) = 10$, $Area(ADN) = 37$