Math 125, Sections F and G, Spring 2007, Solutions to Midterm II

1. Find the following definite integrals. (5 points each)

   (a) $\int_1^e x \ln x dx$
   Using integration by parts:

$$u = \ln x \qquad dv = x dx$$
$$du = \frac{1}{x} dx \qquad v = \frac{1}{2}x^2$$

   So

$$\int_1^e x \ln x dx = \frac{1}{2}x^2 \ln x\Big|_1^e - \int_1^e \frac{1}{2} x dx = \frac{1}{2}x^2 \ln x\Big|_1^e - \frac{1}{4}x^2\Big|_1^e = (\frac{e^2}{2} - 0) - (\frac{e^2}{4} - \frac{1}{4}) = \frac{e^2+1}{4}$$

   (b) $\int_0^1 x^2\sqrt{1-x^2} dx$
   Using trig substitution:

$$x = \sin\theta \qquad dx = \cos\theta d\theta$$

   Changing the limits:

$$x = 0 = \sin\theta \qquad \theta = 0$$
$$x = 1 = \sin\theta \qquad \theta = \frac{\pi}{2}$$

   The integral becomes

$$\int_0^1 x^2\sqrt{1-x^2}dx = \int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta d\theta$$

   Using the double angle formulas

$$\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta d\theta = \int_0^{\frac{\pi}{2}} \Big(\frac{\sin 2\theta}{2}\Big)^2 d\theta = \int_0^{\frac{\pi}{2}} \Big(\frac{1-\cos 4\theta}{8}\Big) d\theta = \frac{\theta}{8} - \frac{\sin 4\theta}{8}\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{16}$$

2. Answer the following questions. (5+5 points)

   (a) Does the improper integral

$$\int_5^\infty \frac{x-2}{x^2-6x+9}dx$$

   converge or diverge?
   It diverges. You can integrate it or use comparison.
   Integrating: Since $x^2-6x+9 = (x-3)^2$ we need to do partial fractions:

$$\frac{x-2}{x^2-6x+9} = \frac{A}{x-3} + \frac{B}{(x-3)^2} = \frac{A(x-3)+B}{(x-3)^2}$$

   So

$$x - 2 = A(x-3) + B$$

   Sunstituting $x = 3$ gives us $B = 1$. Sunstituting any other vale for x for exampe $x = 4$ gives $2 = A + 1$ so $A = 1$. Therefore:

$$\lim_{t\to\infty}\int_5^t \frac{1}{x-3} + \frac{1}{(x-3)^2}dx = \lim_{t\to\infty}\Big(\ln|x-3| - \frac{1}{x-3}\Big|_5^t\Big) = \infty$$

so the integral diverges.

You can also compare it to the integral

$$\int_5^\infty \frac{1}{x} dx$$

which we know is divergent because $p = 1$. Since

$$9 \le 4x$$

when $x \ge 5$

$$x^2 - 6x + 9 \le x^2 - 2x$$

so

$$\frac{1}{x} \le \frac{x-2}{x^2 - 6x + 9}$$

By comparison, since the smaller integral diverges, so does the larger one.

(b) Find the average value of the function $f(x) = x\sqrt[3]{5x+3}$ over the interval $[0, 1]$. Using the average value formula we have

$$A = \frac{1}{1-0}\int_0^1 x\sqrt[3]{5x+3}dx = \int_0^1 x\sqrt[3]{5x+3}dx$$

Because of the term $\sqrt[3]{5x+3}$ we try u substitution with $u = \sqrt[3]{5x+3}$.

$$u^3 = 5x + 3 \qquad\qquad 3u^2 du = 5dx$$

$$\text{when } x = 0 \text{ we get } u^3 = 3 \text{ so } u = \sqrt[3]{3}$$

$$\text{when } x = 1 \text{ we get } u^3 = 8 \text{ so } u = \sqrt[3]{8} = 2$$

Therefore

$$\int_0^1 x\sqrt[3]{5x+3}dx = \int_{\sqrt[3]{3}}^2 \frac{u^3-3}{5} u\frac{du}{5} = \frac{1}{25}\int_{\sqrt[3]{3}}^2 (u^4 - 3u)du$$

$$= \left(\frac{u^5}{5} - \frac{3u^2}{2}\right)\Big|_{\sqrt[3]{3}}^2 = \left(\frac{2^5}{5} - \frac{3\times 2^2}{2}\right) - \left(\frac{3^{5/3}}{5} - \frac{3\times 3^{2/3}}{2}\right) = -\frac{14}{5} + \frac{9}{10}3^{2/3}$$

3. Find the length of the parametric curve

$$x = t^3 + 1 \qquad\qquad y = t^2 - 1 \qquad\qquad 0 \le t \le 2$$

We use the formula

$$S = \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^2 \sqrt{(3t^2)^2 + (2t)^2} dt = \int_0^2 \sqrt{9t^4 + 4t^2} dt = \int_0^2 t\sqrt{9t^2+4} dt$$

Let $u = 9t^2 + 4$ then $du = 18tdt$ so

$$\int_0^2 t\sqrt{9t^2+4}dt = \int_4^{40} \sqrt{u}\frac{du}{18} = \frac{2u^{3/2}}{3\times 18}\Big|_4^{40} = \frac{40^{3/2} - 4^{3/2}}{27} = \frac{40^{3/2} - 8}{27}$$

4. A tank whose lowest point is 10 feet above the ground has the shape of a cup obtained by rotating the parabola $x^2 = 5y, -5 \leq x \leq 5$, around the y-axis. The units on the coordinate axes are in feet. How much work is done in filling this tank with oil of density 50 pounds per cubic feet if the oil is pumped in from ground level?

$$\Delta\text{Work} = \text{distance} \times \Delta\text{Weight} = \text{distance} \times \rho \times \Delta\text{Volume} = \text{distance} \times \rho \times \pi \times \text{radius}^2 \times \Delta y$$

When we make a slice at y the distance is $10 + y$ and the radius is x. But since we will integrate with respect to y (because of Δy) we use $\text{radius}^2 = x^2 = 5y$. So

$$\text{distance} \times \rho \times \pi \times \text{radius}^2 \times \Delta y = (10 + y) \times \rho \times \pi \times 5y \times \Delta y$$

For the limites of integration we need the value for y when $x = \pm 5$ which gives $y = 5$. Therefore the total work done is

$$\int_0^5 \rho\pi(10 + y)5y dy = \rho\pi \int_0^5 (50y + 5y^2)dy = \rho\pi(25y^2 + 5\frac{y^3}{3})\Big|_0^5 = \rho\pi(625 + \frac{625}{3}) = 4\rho\pi\frac{625}{3}$$