Math 125, Sections E and F, Fall 2011, Solutions to Midterm II

1. Evaluate the following indefinite integrals.

   (a) u-sub

$$u = \sin(3t) \qquad\qquad du = 3\cos(3t)dt$$

$$\int \frac{\cos(3t)}{1+\sin^2(3t)}dt = \frac{1}{3}\int \frac{du}{1+u^2} = \frac{1}{3}\tan^{-1}u + C = \frac{1}{3}\tan^{-1}(\sin(3t)) + C$$

   (b) Partial fractions, Case III

$$\int \frac{2x^2+3x+4}{(x-1)(x^2+9)}dx = \frac{9}{10}\int \frac{1}{x-1}dx + \frac{11}{10}\int \frac{x}{x^2+9}dx + \frac{41}{10}\int \frac{1}{x^2+9}dx$$

$$= \frac{9}{10}\ln(|x-1| + \frac{11}{20}\ln|x^2+9| + \frac{41}{30}\tan^{-1}\left(\frac{x}{3}\right) + C$$

2. Evaluate the following integrals.

   (a) Inverse trig substitution

$$x = 2\sec\theta \qquad\qquad dx = 2\sec\theta\tan\theta$$

$$\int_2^4 \frac{\sqrt{x^2-4}}{x^3}dx = \int_0^{\pi/3} \frac{2\tan\theta}{8\sec^3\theta}2\sec\theta\tan\theta d\theta = \frac{1}{2}\int_0^{\pi/3}\sin^2\theta d\theta = \frac{1}{4}\int_0^{\pi/3} 1 - \cos 2\theta d\theta = \frac{1}{4}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$$

   (b) Integration is by parts

$$u = \ln x \qquad du = \frac{1}{x}dx \qquad dv = x^{-3}dx \qquad v = \frac{x^{-2}}{-2}$$

$$\int_1^\infty \frac{\ln x}{x^3}dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^3}dx = \lim_{t\to\infty}\left[\left.\frac{\ln x}{-2x^2}\right|_1^t - \int_1^t \frac{x^{-3}}{-2}dx\right] = \lim_{t\to\infty}\left[-\frac{\ln t}{2t^2} - \frac{4}{t^2} + \frac{1}{4}\right] = \frac{1}{4}$$

3. (a) Use Simpson's Rule with $n = 6$ to approximate the integral

$$\int_0^1 \frac{e^x}{1+e^x}dx.$$

$$S_6 = \frac{1/6}{3}\left[f(0) + 4f(1/6) + 2f(2/6) + 4f(3/6) + 2f(4/6) + 4f(5/6) + f(1)\right] \approx 0.6201149$$

(b) Evaluate the same integral exactly and find the percentage error in your Simpson's Rule approximation. Percentage error$=\dfrac{\text{error}}{\text{actual value}} \times 100$ percent.

$$\int_0^1 \frac{e^x}{1+e^x}dx = \ln(1+e^x)\Big|_0^1 = \ln\left(\frac{e+1}{2}\right) \approx 0.6201145$$

So the percentage error is approximately

$$\frac{0.6201145 - 0.6201149}{0.6201145} \times 100 = -0.0000645\%$$

4. A tank is formed by rotating the parabola $y = x^2 + 1$ about the y-axis. Water is pumped into this tank from ground level at $y = 0$.

(a) (7 points) Find the depth of the water h at its deepest point after $24500\pi/3$ Joules of work has been done. The density of water is 1000 kilograms per cubic meter and the acceleration due to gravity is approximately 9.8 meters per second squared.

The work integral is:

$$\frac{24500\pi}{3} = \int_1^{1+h} \pi(y-1)y(9.8)(1000)dy$$

Simplifying and evaluating the integral we get

$$\frac{5}{6} = \frac{y^3}{3} - \frac{y^2}{2}\Big|_1^{1+h}$$

or

$$5 = 3h^2 + 2h^3$$

At this point you can see that $h = 1$.

(b) (3 points) What is the mass of the water in the tank after this work?

$$\text{Mass} = \int_1^{1+1} \pi(y-1)(1000)dy = 2000\pi\text{kg}$$