Math 125, Sections F and G, Autumn 2012, Solutions to Midterm II

1. Evaluate the following integrals.

   (a) $\int x\ln(x+4)\quad dx$
   Integration by parts:

$$u = \ln(4+x) \qquad dv = xdx$$

$$du = \frac{1}{x+4}dx \qquad v = \frac{x^2}{2}$$

   So,

$$\int x\ln(x+4)\quad dx = \frac{x^2\ln(x+4)}{2} - \frac{1}{2}\int \frac{x^2}{4+x}dx$$

   now use long division and integrate

$$\int x\ln(x+4)\quad dx = \frac{x^2\ln(x+4)}{2} - \frac{1}{2}\int x-4+\frac{16}{4+x}dx = \frac{x^2\ln(x+4)}{2} - \frac{x^2}{4} + 2x - 8\ln|x+4| + C$$

   (b) $\int_{1/2}^{1} \frac{dx}{x^3\sqrt{4x^2-1}}$
   Inverse trig substitution: $2x = \sec\theta$ and $2dx = \sec\theta\tan\theta d\theta$. So,

$$\int_{1/2}^{1} \frac{dx}{x^3\sqrt{4x^2-1}} = \frac{8}{2}\int_0^{\frac{\pi}{3}} \frac{\sec\theta\tan\theta}{\sec^3\theta\tan\theta}d\theta = 4\int_0^{\frac{\pi}{3}} \cos^2\theta d\theta$$

   using the double angle formula

$$= 2\int_0^{\frac{\pi}{3}} 1+\cos 2\theta d\theta = 2\theta + \sin 2\theta\Big|_0^{\pi/3} = \frac{2\pi}{3} + \frac{\sqrt{3}}{2}.$$

2. For the improper integral

$$\int_1^{\infty} \frac{12x+6}{x^3+5x^2+6x}dx$$

   (a) Determine if it converges or diverges using the comparison theorem. Give a brief explanation.
   The integral coverges.

$$\frac{12x+6}{x^3+5x^2+6x} < \frac{12x+6}{x^3} = \frac{12}{x^2} + \frac{6}{x^3}$$

   and

$$\int_1^{\infty} \frac{12}{x^2} + \frac{6}{x^3}dx = 12\int_1^{\infty} \frac{1}{x^2}dx + 6\int_1^{\infty} \frac{1}{x^3}dx = \frac{12}{2-1} + \frac{6}{3-1}$$

   ($p = 2$ and $p = 3$)

   (b) Now evaluate it to see if it diverges or converges. If it converges, what value does it converge to?
   To evaluate the improper integral we use limits

$$\int_1^{\infty} \frac{12x+6}{x^3+5x^2+6x}dx = \lim_{t\to\infty}\int_1^{t} \frac{12x+6}{x^3+5x^2+6x}dx = \lim_{t\to\infty}\int_1^{t} \frac{12x+6}{x(x+2)(x+3)}dx$$

   and then use partial fractions

$$= \lim_{t\to\infty}\int_1^{t} \frac{1}{x} + \frac{9}{x+2} + \frac{-10}{x+3}dx = \lim_{t\to\infty}\left[\ln x + 9\ln(x+2) - 10\ln(x+3)\right]_1^t$$

$$\lim_{t\to\infty}\left[(\ln t + 9\ln(t+2) - 10\ln(t+3)) - (\ln 1 + 9\ln(3) - 10\ln(4))\right] = \lim_{t\to\infty}\left[\ln\frac{t(t+2)^9}{(t+3)^{10}} - \ln\frac{3^9}{4^{10}}\right] = \ln\left(\frac{4^{10}}{3^9}\right).$$

3. This question concerns the length of the curve $y = \left(\frac{2}{3}x\right)^{\frac{3}{2}}$ from the point $(3/2, 1)$ to the point $(6, 8)$.

   (a) (4 points) Set up two integrals for the length of the curve. One should end in dx, the other in dy.

$$L = \int_{3/2}^{6} \sqrt{1 + \frac{2}{3}x} \;\; dx = \int_{1}^{8} \sqrt{1 + y^{-2/3}} \;\; dy$$

   (b) (3 points) Pick one of the integrals and evaluate it to find the exact length of the curve. The first one can be integrated wth substitution $u = 1 + \frac{2}{3}x$ and $du = \frac{2}{3}dx$,

$$L = \int_{3/2}^{6} \sqrt{1 + \frac{2}{3}x} \;\; dx = \frac{3}{2}\int_{2}^{5} \sqrt{u} \;\; du = u^{3/2}\big|_2^5 = 5^{3/2} - 2^{3/2}.$$

   (c) (2 points)Approximate the OTHER integral using Simpson's rule with $n = 6$. Here $f(y) = \sqrt{1 + y^{-1/3}}$ and $\Delta y = \frac{8-1}{6} = \frac{7}{6}$. So,

$$L \approx \frac{7}{6 \cdot 3}\Big[\sqrt{1 + 1^{-2/3}} + 4\sqrt{1 + (13/6)^{-2/3}} + 2\sqrt{1 + (20/6)^{-2/3}} + 4\sqrt{1 + (27/6)^{-2/3}}$$

$$+2\sqrt{1 + (34/6)^{-2/3}} + 4\sqrt{1 + (41/6)^{-2/3}} + \sqrt{1 + 8^{-2/3}}\Big] \approx 8.2554$$

   (d) (1 point) Find the percentage error you made in your approximation in part (c) by comparing with the exact value you found in part (b).

$$\text{Percentage Error} \approx \frac{5^{3/2} - 2^{3/2} - 8.2554}{5^{3/2} - 2^{3/2}} \times 100\% \approx 1.15\%$$

4. A tank is formed by rotating the parabola $y = \frac{1}{5}x^2$ between $x = 0$ and $x = 5$ about the y-axis. The units are in meters. Initially, it is full of water of density 1000 kg/m^3. It is shown partially full in the picture below. The depth of the water in the tank is measured at its deepest point with the meter stick show in the picture.

   (a) (8 points) Water is pumped out from an outlet 2 meters above its top until the depth of the remaining water is 1 meter. Find the work done.

$$\text{Work} = \int_{1}^{5} \pi(5y)(9800)(7 - y)dy = \frac{6737500}{3}\pi \text{ Joules}$$

   (b) (2 points) Find the weight of the water left in the tank.

$$\text{Weight} = \int_{0}^{1} \pi(5y)(9800)dy = 24500\pi \text{ Newtons}$$